### ANALYTIC IN THE UNIT BALL FUNCTIONS OF BOUNDED L-INDEX IN DIRECTION

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#### Abstract

We propose a generalisation of analytic in a domain function of bounded index, which was introduced by J. G. Krishna and S. M. Shah [14]. In fact, analytic in the unit ball function of bounded index by Krishna and Shah is an entire function. Our approach allows us to explore properties of analytic in the unit ball functions.

We proved the necessary and sufficient conditions of bounded L-index in direction for analytic functions. As a result, they are applied to study partial differential equations and get sufficient conditions of bounded L-index in direction for analytic solutions. Finally, we estimated growth for these functions.

**Keywords:** analytic function, unit ball, bounded L-index in direction, growth estimates, partial differential equation, several complex variables

MSC[2010] 32A10, 32A17, 35B08

#### 1 Introduction

B. Lepson [19] introduced a class of entire functions of bounded index. He raised the problem to characterise entire functions of bounded index. An entire function f is said to be of bounded index if there exists an integer N > 0 that

$$(\forall z \in \mathbb{C})(\forall n \in \{0, 1, 2, \ldots\}) \colon \frac{|f^{(n)}(z)|}{n!} \le \max \left\{ |f(z)|, \frac{|f^{(j)}(z)|}{j!} \colon 1 \le j \le N \right\}. \tag{1}$$

The least such integer N is called the index of f.

Afterwards, S. Shah [21] and W. Hayman [13] independently proved that every entire function of bounded index is a function of exponential type. Namely, its growth is at most the first order and normal type. Further, W. Hayman showed that an entire function is of bounded value distribution if and only if its derivative is of bounded index. An entire function f is said to be of bounded

value distribution if for every r > 0 there exists a fixed integer p(r) > 0 such that the equation f(z) = w has never more than p(r) roots in any disc of radius r and for any  $w \in \mathbb{C}$ . The functions of bounded index have been used in the theory value distribution and differential equations (see bibliography in [21]).

- T. Lakshminarasimhan [18] generalised a bounded index. He introduced entire functions of L-bounded index, where L(r) is a positive continuous slowly increasing function. D. Somasundaram and R. Thamizharasi [25]-[26] continued his investigations of entire functions of L-bounded index. They studied growth properties and characterisations of these functions.
- B. C. Chakraborty, Rita Chanda and Tapas Kumar Samanta [10]-[12] introduced bounded index and L-bounded index for entire functions in  $\mathbb{C}^n$ . They found a necessary and sufficient condition for an entire function to be of L-bounded index and proved some interesting properties.
- J. Gopala Krishna and S. M. Shah [14] studied of the existence and analytic continuation of the local solutions of partial differential equations. They introduced an analytic in a domain (a nonempty connected open set)  $\Omega \subset \mathbb{C}^n$   $(n \in \mathbb{N})$  function of bounded index for  $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$ . Namely, let  $\Omega_+ = \{z = (z_1, \ldots, z_n) \in \Omega \colon z_j > 0 \ (j \in \{1, \ldots, n\})\}$ , that is a subset of all points of  $\Omega$  with positive real coordinates. We say that an analytic in  $\Omega$  function F is a function of bounded index (Krishna-Shah bounded index or  $F \in \mathcal{B}(\Omega, \alpha)$ ) for  $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Omega_+$  in domain  $\Omega$  if and only if there exists  $N = N(\alpha, F) = (N_1, \ldots, N_n) \in \mathbb{Z}_+^n$  such that inequality

$$\alpha^m T_m(z) \le \max\{\alpha^p T_p(z) \colon p \le N\},$$

is valid for all  $z \in \Omega$  and for every  $m \in \mathbb{Z}_+^n$ , where  $\alpha^m = \alpha_1^{m_1} \cdots \alpha_n^{m_n}$ ,  $T_m(z) = |F^{(m)}(z)|/m!$ ,  $F^{(m)}(z) = \frac{\partial^{\|m\|}F}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$  be  $\|m\|$ -th partial derivative of F,  $F^{(0,\dots,0)} = F$ ,  $m! = m_1! \cdots m_n!$ ,  $\|m\| = m_1 + \dots + m_n$ ,  $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$ .

For entire functions in two variables, M. Salmassi [20] generalised bounded index and proved three criteria of index boundedness. Besides, he researched a system of partial differential equations and found conditions of bounded index for entire solutions.

To consider the functions of nonexponential type A. D. Kuzyk and M. M. Sheremeta [17] introduced a bounded l-index, replacing  $\frac{|f^{(p)}(z)|}{p!}$  on  $\frac{|f^{(p)}(z)|}{p!l^p(|z|)}$  in (1), where  $l: \mathbb{R}_+ \to \mathbb{R}_+$  is a continuous function. Besides, they proved that growth of entire function of L-bounded index is not higher than a normal type and first order.

Afterwards, S. M. Strochyk and M. M. Sheremeta [27] considered bounded l-index for functions, that are analytic in a disc. Later T. O. Banakh, V. O. Kushnir and M. M. Sheremeta generalised this term for analytic in arbitrary complex domain  $G \subset \mathbb{C}$  functions ([1], [15] – [16]). Yu. S. Trukhan and M. M. Sheremeta got sufficient conditions of bounded l-index for infinite products, which are analytic in the unit disc. In particular, they researched Blaschke product and Naftalevich-Tsuji product ([23], [28] – [32]).

M. T. Bordulyak and M. M. Sheremeta ([8] - [9]) defined a function of bounded L-index

in joint variables, where  $\mathbf{L} = \mathbf{L}(z) = (l_1(z_1), \dots, l_n(z_n)), \ l_j(z_j)$  are positive continuous functions,  $j \in \{1, \dots, n\}$ . If  $\mathbf{L}(z) \equiv \left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right)$  and  $\Omega = C^n$  then a Bordulyak-Sheremeta's definition matches with a Krishna - Shah's definition. If n = 2 and  $\mathbf{L}(z) \equiv (1, 1)$  then a Bordulyak-Sheremeta's definition matches with a Salmassi's definition [20].

Methods for investigation of analytic functions in  $\mathbb{C}^n$  are divided into several groups. One group is based on the study of function F as analytic in each variable separately. Other methods arise in the study of slice function that is analytic functions of one variable  $g(\tau) = F(a+b\tau), \tau \in \mathbb{C}$ . This is a restriction of the analytic function F to arbitrary complex lines  $\{z = a + b\tau : \tau \in \mathbb{C}\}, a, b \in \mathbb{C}^n$ .

Using the first approach, M. T. Bordulyak and M. M. Sheremeta [8] proved many properties and criteria of bounded  $\mathbf{L}$ -index for entire functions in  $\mathbb{C}^n$ . They got sufficient conditions of bounded  $\mathbf{L}$ -index for entire solutions of some systems of partial differential equations. However, this approach did not allow to find an equivalent to a criterion of bounded  $\mathbf{L}$ -index by the estimate of the logarithmic derivative outside zero set. In particular, efforts to explore  $\mathbf{L}$ -index boundedness for some important classes of entire functions (for example, infinite products with "plane" zeros) were unsuccessful by technical difficulties.

For the reasons given above, there was a natural problem to consider and to explore an entire in  $\mathbb{C}^n$  function of bounded L-index by a second approach.

Applying this method, we proposed a new approach to introduce an entire in  $\mathbb{C}^n$  function of bounded L-index in direction [3] – [7]. In contrast to the approach proposed by M.T. Bordulyak and M. M. Sheremeta, our definition is based on directional derivative. It allowed to generalize more results from  $\mathbb{C}$  to  $\mathbb{C}^n$  and find new assertions because a definition contains a directional derivative and it has influence on the L-index.

This success gives possibility of generalisation of bounded L-index in direction for analytic in a ball functions. Besides, analytic in a domain function of bounded index by Krishna and Shah is an entire function. It follows from necessary condition of l-index boundedness for analytic in the unit disc function ([22],Th.3.3, p.71):  $\int_0^r l(t)dt \to \infty$  as  $r \to 1$ . In this paper, we proved criteria of L-index boundedness in direction, which describe a maximum modulus estimate on a larger circle by maximum modulus on a smaller circle, an analogue of Hayman Theorem, a maximum modulus estimate on circle by minimum modulus on circle, an estimate of logarithmic directional derivative outside zero set and an estimate of counting function of zeros. They helped to get conditions on partial differential equation which provide bounded L-index in direction for analytic solutions. Finally, we describe the growth of analytic in  $\mathbb{B}_n$  function of bounded L-index in direction.

**Remark.** We investigate analytic functions in the unit ball instead the ball of arbitrary radius.

### 2 Main definition and properties functions of bounded *L*-index in direction

Let  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  be a given direction,  $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ ,  $\overline{\mathbb{B}}_n = \{z \in \mathbb{C}^n : |z| \le 1\}$ ,  $L : \mathbb{B}_n \to \mathbb{R}_+$  be a continuous function that for all  $z \in \mathbb{B}_n$ 

$$L(z) > \frac{\beta |\mathbf{b}|}{1 - |z|}, \ \beta = \text{const} > 1, \mathbf{b} \neq 0.$$
 (2)

For a given  $z \in \mathbb{B}_n$  we denote  $S_z = \{t \in \mathbb{C} : z + t\mathbf{b} \in \mathbb{B}_n\}.$ 

**Remark 1.** Notice that if  $\eta \in [0, \beta]$ ,  $z \in \mathbb{B}_n$ ,  $z + t_0 \mathbf{b} \in \mathbb{B}_n$  and  $|t - t_0| \le \frac{\eta}{L(z + t_0 \mathbf{b})}$  then  $z + t \mathbf{b} \in \mathbb{B}_n$ . Indeed, we have  $|z + t \mathbf{b}| = |z + t_0 \mathbf{b} + (t - t_0) \mathbf{b}| \le |z + t_0 \mathbf{b}| + |(t - t_0) \mathbf{b}| \le |z + t_0 \mathbf{b}| + \frac{\eta |\mathbf{b}|}{L(z + t_0 \mathbf{b})} < |z + t_0 \mathbf{b}| + \frac{\beta |\mathbf{b}|}{\frac{\beta |\mathbf{b}|}{1 - |z + t_0 \mathbf{b}|}} = 1$ .

Analytic in  $\mathbb{B}_n$  function F(z) is called a function of bounded L-index in a direction  $\mathbf{b} \in \mathbb{C}^n$ , if there exists  $m_0 \in \mathbb{Z}_+$  that for every  $m \in \mathbb{Z}_+$  and every  $z \in \mathbb{B}_n$  the following inequality is valid

$$\frac{1}{m!L^m(z)} \left| \frac{\partial^m F(z)}{\partial \mathbf{b}^m} \right| \le \max \left\{ \frac{1}{k!L^k(z)} \left| \frac{\partial^k F(z)}{\partial \mathbf{b}^k} \right| : 0 \le k \le m_0 \right\},\tag{3}$$

where 
$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z)$$
,  $\frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} \ F, \ \overline{\mathbf{b}} \rangle$ ,  $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}} \left( \frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}} \right)$ ,  $k \ge 2$ .

The least such integer  $m_0 = m_0(\mathbf{b})$  is called the *L*-index in direction  $\mathbf{b}$  of the analytic function F(z) and is denoted by  $N_{\mathbf{b}}(F, L) = m_0$ . If n = 1,  $\mathbf{b} = 1$ , L = l, F = f, then  $N(f, l) \equiv N_1(f, l)$  is called the *l*-index of function f.

In the case n = 1 and  $\mathbf{b} = 1$  we have definition of analytic in the unit disc function of bounded l-index [27].

Now we state several lemmas that contain the basic properties of analytic in the unit ball functions of bounded L-index in direction. Let  $l_z(t) = L(z + t\mathbf{b})$ ,  $g_z(t) = F(z + t\mathbf{b})$  for given  $z \in \mathbb{C}^n$ .

**Lemma 1.** If F(z) is an analytic in  $\mathbb{B}_n$  function of bounded L-index  $N_{\mathbf{b}}(F, L)$  in direction  $\mathbf{b} \in \mathbb{C}^n$ , then for every  $z^0 \in \mathbb{B}_n$  the analytic function  $g_{z^0}(t)$ ,  $t \in S_{z^0}$ , is of bounded  $l_{z^0}$ -index and  $N(g_{z^0}, l_{z^0}) \leq N_{\mathbf{b}}(F, L)$ .

*Proof.* Let  $z^0 \in \mathbb{B}_n$  be a fixed point and  $g(t) \equiv g_{z^0}(t), \ l(t) \equiv l_{z^0}(t)$ . Since for every  $p \in \mathbb{N}$ 

$$g^{(p)}(t) = \frac{\partial^p F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^p},\tag{4}$$

then by the definition of bounded L-index in direction **b** for all  $t \in S_{z^0}$  and for all  $p \in \mathbb{Z}_+$  we obtain

$$\frac{|g^{(p)}(t)|}{p!l^p(t)} = \frac{1}{p!L^p(z^0 + t\mathbf{b})} \left| \frac{\partial^p F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^p} \right| \leq \max \left\{ \frac{1}{k!L^k(z^0 + t\mathbf{b})} \left| \frac{\partial^k F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^k} \right| : \right.$$

$$0 \le k \le N_{\mathbf{b}}(F, L) \} = \max \left\{ \frac{|g^{(k)}(t)|}{k! l^{k}(t)} : 0 \le k \le N_{\mathbf{b}}(F, L) \right\}.$$

From here, g(t) is a function of bounded l-index and  $N(g, l) \leq N_{\mathbf{b}}(F, L)$ . Lemma 1 is proved.  $\square$ 

An equation (4) implies a following proposition.

**Lemma 2.** If F(z) is an analytic in  $\mathbb{B}_n$  function of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  then  $N_{\mathbf{b}}(F, L) = \max \{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{B}_n\}.$ 

However, maximum can be calculated on the subset A with points  $z^0$ , which has property  $\{z^0 + t\mathbf{b} : t \in S_{z^0}, z^0 \in A\} = \mathbb{B}_n$ . So the following assertion is valid.

**Lemma 3.** If F(z) is an analytic in  $\mathbb{B}_n$  function of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  and  $j_0$  is chosen with  $b_{j_0} \neq 0$  then  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{C}^n, z_{j_0}^0 = 0\}$  and if  $\sum_{j=1}^n b_j \| \neq 0$  then  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{C}^n, \sum_{j=1}^n z_j^0 = 0\}$ .

*Proof.* We prove that for every  $z \in \mathbb{B}_n$  there exist  $z^0 \in \mathbb{C}^n$  and  $t \in S_{z^0}$  with  $z = z^0 + t\mathbf{b}$  and  $z^0_{j_0} = 0$ . Put  $t = z_{j_0}/b_{j_0}$ ,  $z^0_j = z_j - t\mathbf{b}_j$ ,  $j \in \{1, 2, ..., n\}$ . Clearly,  $z^0_{j_0} = 0$  for this choice.

However, a point  $z^0$  may not be contained in  $\mathbb{B}_n$ . But there exists  $t \in \mathbb{C}$  that  $z^0 + t\mathbf{b} \in \mathbb{B}_n$ . Let  $z^0 \notin \mathbb{B}_n$  and  $|z| = R_1 < 1$ . Therefore,  $|z^0 + t\mathbf{b}| = |z - \frac{z_{j_0}}{b_{j_0}}\mathbf{b} + t\mathbf{b}| = |z + (t - \frac{z_{j_0}}{b_{j_0}})\mathbf{b}| \le |z| + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| \le R_1 + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| < 1$ . Thus,  $|t - \frac{z_{j_0}}{b_{j_0}}| < \frac{1 - R_1}{|\mathbf{b}|}$ .

In second part we prove for every  $z \in \mathbb{B}_n$  there exist  $z^0 \in \mathbb{C}^n$  and  $t \in S_{z^0}$  that  $z = z^0 + t\mathbf{b}$  and  $\sum_{j=1}^n z_j^0 = 0$ . Put  $t = \frac{\sum_{j=1}^n z_j}{\sum_{j=1}^n b_j}$  and  $z_j^0 = z_j - t\mathbf{b}_j$ ,  $1 \le j \le n$ . Thus, the following equality is valid  $\sum_{j=1}^n z_j^0 = \sum_{j=1}^n (z_j - tb_j) = \sum_{j=1}^n z_j - \sum_{j=1}^n b_j t = 0$ .

Lemma 3 is proved.  $\Box$ 

Note that for a given  $z \in \mathbb{B}_n$  we can pick uniquely  $z^0 \in \mathbb{C}^n$  and  $t \in S_{z^0}$  such that  $\sum_{j=1}^n z_j^0 = 0$  and  $z = z^0 + t\mathbf{b}$ .

**Remark 2.** If for some  $z^0 \in \mathbb{C}^n$   $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\} \cap \mathbb{B}_n = \emptyset$  then we put  $N(g_{z^0}, l_{z^0}) = 0$ .

Lemmas 1–3 imply the following proposition.

**Theorem 1.** An analytic in  $\mathbb{B}_n$  function F(z) is a function of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exists number M > 0 such, that for every  $z^0 \in \mathbb{B}_n$  function  $g_{z^0}(t)$  is of bounded  $l_{z^0}$ -index with  $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ , as a function of one variable  $t \in S_{z^0}$ , and  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{B}_n\}.$ 

*Proof.* Necessity follows from Lemma 1.

We prove sufficiency.

Since  $N(g_{z^0}, l_{z^0}) \leq M$  there exists  $\max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{B}_n\}$ . We denote this maximum by  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{B}_n\} < \infty$ . Suppose that  $N_{\mathbf{b}}(F)$  is not L-index in direction  $\mathbf{b}$  of function F(z). So there exists  $n^* > N_{\mathbf{b}}(F, L)$  and  $z^* \in \mathbb{B}_n$ 

$$\frac{1}{n^*!L^{n^*}(z^*)} \frac{\left|\partial^{n^*}F(z^*)\right|}{\partial \mathbf{b}^{n^*}} > \max\left\{\frac{1}{k!L^k(z^*)} \frac{\left|\partial^k F(z^*)\right|}{\partial \mathbf{b}^k}, 0 \le k \le N_{\mathbf{b}}(F, L)\right\}.$$
 (5)

But we have  $g_{z^0}(t) = F(z^0 + t\mathbf{b}), g_{z^0}^{(p)}(t) = \frac{\partial^p F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^p}$ . We can rewrite (5) as

$$\frac{\left|g_{z^*}^{(n^*)}(0)\right|}{n^*!l_{z^*}^{n^*}(0)} > \max\left\{\frac{\left|g_{z^*}^{(k)}(0)\right|}{k!l_{z^*}^{k}(0)} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\}.$$

It contradicts that all  $l_{z^0}$ -indices  $N(g_{z_0}, l_{z^0})$  are bounded by number  $N_{\mathbf{b}}(F)$ . Thus  $N_{\mathbf{b}}(F)$  is L-index in direction  $\mathbf{b}$  of function F(z). Theorem 1 is proved.

From Lemma 3 the following condition is enough in Theorem 1: there exists  $M < +\infty$  that an inequality holds  $N(g_{z^0}, l_{z^0}) \leq M$  for every  $z^0 \in \mathbb{C}^n$  with  $\sum_{j=1}^n z_j^0 = 0$ .

Since Lemma 3 and 1 there is a natural question: what is the least set A that the following equality is valid  $N_{\mathbf{b}}(F, L) = \max_{z^0 \in A} N(g_{z^0}, l_{z^0})$ .

Below we prove propositions that give a partial answer to this question. A solution is partial because it is unknown whether our sets are the least which satisfy the mentioned equality.

**Theorem 2.** Let  $\mathbf{b} \in \mathbb{C}^n$  be a given direction,  $A_0$  be an arbitrary set in  $\mathbb{C}^n$  with  $\{z + t\mathbf{b} : t \in S_z, z \in A_0\} = \mathbb{B}_n$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exists M > 0 that for all  $z^0 \in A_0$  function  $g_{z^0}(t)$  is of bounded  $l_{z^0}$ -index  $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ , as a function of variable  $t \in S_{z^0}$ . And  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$ .

Proof. By Theorem 1, analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exists number M > 0 such that for every  $z^0 \in \mathbb{B}_n$  function  $g_{z^0}(t)$  is of bounded  $l_{z^0}$ -index  $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ , as a function of variable  $t \in S_{z^0}$ . But for every  $z^0 + t\mathbf{b}$  by properties of set  $A_0$  there exist  $\tilde{z}^0 \in A_0$  and  $\tilde{t} \in \mathbb{B}_{\tilde{z}^0}$ 

$$z^0 + t\mathbf{b} = \widetilde{z}^0 + \widetilde{t}\mathbf{b}.$$

For all  $p \in \mathbb{Z}_+$  we have

$$(g_{z_0}(t))^{(p)} = (g_{\widetilde{z}_0}(\widetilde{t}))^{(p)}.$$

But  $\tilde{t}$  is dependent of t. Therefore, a condition  $g_{z^0}(t)$  is of bounded  $l_{z^0}$ -index for all  $z^0 \in \mathbb{B}_n$  is equivalent to a condition  $g_{\tilde{z}^0}(t)$  is of bounded  $l_{\tilde{z}^0}$ -index for all  $\tilde{z}^0 \in A_0$ .

Remark 3. An intersection of arbitrary hyperplane  $H = \{z \in \mathbb{C}^n : \langle z, c \rangle = 1\}$  and set  $\mathbb{B}_n^{\mathbf{b}} = \{z + \frac{1 - \langle z, \mathbf{c} \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} \colon z \in \mathbb{B}_n\}$ , where  $\langle \mathbf{b}, c \rangle \neq 0$ , satisfies conditions of Theorem 2.

We prove that for every  $w \in \mathbb{B}_n$  there exist  $z \in H \cap \mathbb{B}_n^{\mathbf{b}}$  and  $t \in \mathbb{C}$  such that  $w = z + t\mathbf{b}$ . Choosing  $z = w + \frac{1 - \langle w, c \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} \in H \cap \mathbb{B}_n^{\mathbf{b}}$ ,  $t = \frac{\langle w, c \rangle - 1}{\langle \mathbf{b}, c \rangle}$ , we obtain

$$z + t\mathbf{b} = w + \frac{1 - \langle w, c \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} + \frac{\langle w, c \rangle - 1}{\langle \mathbf{b}, c \rangle} \mathbf{b} = w.$$

**Theorem 3.** Let A be an everywhere dense set in  $\mathbb{B}_n$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exists number M > 0 that for every  $z^0 \in A$  function  $g_{z^0}(t)$  is of bounded  $l_{z^0}$ -index  $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ , as a function of  $t \in S_{z^0}$ , and  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$ .

*Proof.* The necessity follows from Theorem 1 (in this theorem same condition is satisfied for all  $z^0 \in \mathbb{B}_n$ , and we need this condition for all  $z^0 \in A$ , that  $\overline{A} \cap \mathbb{B}_n = \mathbb{B}_n$ ).

Now we prove a sufficiency. Since A has been everywhere dense in  $\mathbb{B}_n$ , for every  $z^0 \in \mathbb{B}_n$  there exists a sequence  $(z^m)$ , that  $z^{(m)} \to z^0$  as  $m \to +\infty$  and  $z^{(m)} \in A$  for all  $m \in \mathbb{N}$ . But  $F(z+t\mathbf{b})$  is of bounded  $l_z$ -index for all  $z \in \overline{A} \cap \mathbb{B}_n$  as a function of t. Therefore, by bounded  $l_z$ -index there exists M > 0 that for all  $z \in A$ ,  $t \in \mathbb{C}$ ,  $p \in \mathbb{Z}_+$ 

$$\frac{|g_z^{(p)}(t)|}{p!l^p(t)} \le \max\left\{\frac{|g_z^{(k)}(t)|}{k!l_z^k(t)} : 0 \le k \le M\right\}.$$

After substitution instead of z a sequence  $z^{(m)} \in A$  and  $z^{(m)} \to z^0$ , for each  $m \in \mathbb{N}$  the following inequality holds

$$\frac{|g_{z^m}^{(p)}(t)|}{p!l_{z^m}^p(t)} \le \max\left\{\frac{|g_{z^m}^{(k)}(t)|}{k!l_{z^m}^k(t)} : 0 \le k \le M\right\}$$

In other words, we have

$$\frac{1}{p!L^{p}(z^{m}+t\mathbf{b})} \left| \frac{\partial^{p} F(z^{m}+t\mathbf{b})}{\partial \mathbf{b}^{p}} \right| \leq \max \left\{ \frac{1}{k!L^{k}(z^{m}+t\mathbf{b})} \left| \frac{\partial^{k} F(z^{m}+t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : 0 \leq k \leq M \right\}.$$
(6)

But F is an analytic in  $\mathbb{B}_n$  function and L is a positive continuous. In (6) we calculate a limit  $m \to +\infty$   $(z^m \to z^0)$ . We have that for all  $z^0 \in \mathbb{B}_n$ ,  $t \in S_{z^0}$ ,  $m \in \mathbb{Z}_+$ 

$$\frac{1}{p!L^p(z^0 + t\mathbf{b})} \left| \frac{\partial^p F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^p} \right| \le \max \left\{ \frac{1}{k!L^k(z^0 + t\mathbf{b})} \left| \frac{\partial^k F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^k} \right| : 0 \le k \le M \right\}.$$

Since this inequality  $F(z^0 + t\mathbf{b})$  is of bounded  $L(z^0 + t\mathbf{b})$ -index too, as a function of t, for every given  $z^0 \in \mathbb{B}_n$ . Applying Theorem 1 we get a needed conclusion. Theorem 3 is proved.

Since Remark 3 and Theorem 3 the following corollary is true.

Corollary 1. Let  $\mathbf{b} \in \mathbb{C}^n$  be a given direction,  $A_0$  be a set in  $\mathbb{C}^n$  and its closure is  $\overline{A}_0 = \{z \in \mathbb{C}^n : \langle z, c \rangle = 1\} \cap \mathbb{B}^{\mathbf{b}}_n$ , where  $\langle c, \mathbf{b} \rangle \neq 0$ ,  $\mathbb{B}^{\mathbf{b}}_n = \{z + \frac{1 - \langle z, \mathbf{c} \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} : z \in \mathbb{B}_n\}$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exists number M > 0 such that for all  $z^0 \in A_0$  function  $g_{z^0}(t)$  is of bounded  $l_{z^0}$ -index  $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ , as a function of variable  $t \in S_{z^0}$ . And  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$ .

*Proof.* Since Remark 3 in Theorem 2 we can take an arbitrary hyperplane  $B_0 = \{z \in \mathbb{B}^n : \langle z, c \rangle = 1\}$ , where  $\langle c, \mathbf{b} \rangle \neq 0$ . Let  $A_0$  be an everywhere dense set in  $B_0$ ,  $\overline{A_0} = B_0$ . Repeating considerations of Theorem 3, we obtain a needed conclusion.

Indeed, the necessity follows from Theorem 1 (in this theorem same condition is satisfied for all  $z^0 \in \mathbb{C}^n$ , and we need this condition for all  $z^0 \in A_0$ , that  $\overline{A}_0 \cap \mathbb{B}_n = \{z \in \mathbb{B}_n : \langle z, c \rangle = 1\}$ ).

To prove the sufficiency, we use a density of the set  $A_0$ . Obviously, for every  $z^0 \in B_0$  there exists a sequence  $z^{(m)} \to z^0$  and  $z^{(m)} \in A_0$ . But  $g_z(t)$  is of bounded  $l_z$ -index for all  $z \in A_0$  as a function of t. Since conditions of Corollary 1, for some M > 0 and for all  $z \in A_0$ ,  $t \in \mathbb{C}$ ,  $p \in \mathbb{Z}_+$  the following inequality holds

$$\frac{g_z^{(p)}(t)}{p!l_z^p(t)} \le \max\left\{\frac{|g_z^{(k)}(t)|}{k!l_z^k(t)} : 0 \le k \le M\right\}.$$

Substituting an arbitrary sequence  $z^{(m)} \in A$ ,  $z^{(m)} \to z^0$  instead of  $z \in A^0$ , we have

$$\frac{|g_{z^{(m)}}^{(p)}(t)|}{p!l_{z^{(m)}}^{p}(t)} \leq \max\left\{\frac{|g_{z^{(m)}}^{(k)}(t)|}{k!l_{z^{(m)}}^{k}(t):0 \leq k \leq M}\right\},$$

i.e.

$$\frac{1}{L^p(z^{(m)}+t\mathbf{b})}\left|\frac{\partial^p F(z^{(m)}+t\mathbf{b})}{\partial \mathbf{b}^p}\right| \leq \max\left\{\frac{1}{k!L^k(z^{(m)}+t\mathbf{b})}\left|\frac{\partial^k F(z^{(m)}+t\mathbf{b})}{\partial \mathbf{b}^k}\right| : 0 \leq k \leq M\right\}.$$

However, F is an analytic in  $\mathbb{B}_n$  function, L is a positive continuous. So we calculate a limit as  $m \to +\infty$   $(z^m \to z)$ . For all  $z^0 \in B_0$ ,  $t \in S_{z^0}$ ,  $m \in \mathbb{Z}_+$  we have

$$\left| \frac{1}{L^p(z^0 + t\mathbf{b})} \left| \frac{\partial^p F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^p} \right| \le \max \left\{ \frac{1}{k! L^k(z^0 + t\mathbf{b})} \left| \frac{\partial^k F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^k} \right| : 0 \le k \le M \right\}.$$

Therefore,  $F(z^0 + t\mathbf{b})$  is of bounded  $L(z^0 + t\mathbf{b})$ -index as a function of t at each  $z^0 \in B^n$ . By Theorem 3 and Remark 3 F is of bounded L-index in direction  $\mathbf{b}$ .

**Remark 4.** Let  $H = \{z \in \mathbb{C}^n : \langle z, c \rangle = 1\}$ . The condition  $\langle c, \mathbf{b} \rangle \neq 0$  is essential. If  $\langle c, \mathbf{b} \rangle = 0$  then for all  $z^0 \in H$  and for all  $t \in \mathbb{C}$  the point  $z^0 + t\mathbf{b} \in H$  because  $\langle z^0 + t\mathbf{b}, c \rangle = \langle z^0, c \rangle + t\langle \mathbf{b}, c \rangle = 1$ . Thus, this line  $z^0 + t\mathbf{b}$  does not describe points outside a hyperplane H.

We consider  $F(z_1, z_2) = \exp(-z_1^2 + z_2^2)$ ,  $\mathbf{b} = (1, 1)$ , c = (-1, 1). On a hyperplane  $-z_1 + z_2 = 1$  function  $F(z_1, z_2)$  takes a look

$$F(z^0 + t\mathbf{b}) = F(z_1^0 + t, z_2^0 + t) = \exp(-(z_1^0 + t)^2 + (1 + z_1^0 + t)^2) =$$

$$= \exp(1 + 2z_1^0 + 2t).$$

Using definition of l-index boundedness and evaluating corresponding derivatives it is easy to prove that  $\exp(1+2z_1^0+2t)$  is of bounded index with l(t)=1 and N(g,l)=4.

Thus, F is of unbounded index in direction **b**. On the contrary, we assume  $N_{\mathbf{b}}(F) = m$  and calculate directional derivatives

$$\frac{\partial^p F}{\partial \mathbf{b}^p} = 2^p (-z_1 + z_2)^p \exp(-z_1 + z_2), \ p \in \mathbb{N}.$$

By definition of bounded index, an inequality holds  $\forall p \in \mathbb{N} \ \forall z \in \mathbb{C}^n$ 

$$2^{p}|-z_{1}+z_{2}|^{p}|\exp(-z_{1}+z_{2})| \leq \max_{0 \leq k \leq m} 2^{k}|-z_{1}+z_{2}|^{k}|\exp(-z_{1}+z_{2})|.$$
 (7)

Let p > m and  $|-z_1 + z_2| = 2$ . Dividing equation (7) by  $2^p |\exp(-z_1 + z_2)|$ , we get  $2^{2p} \le 2^{2m}$ . It is impossible. Therefore, F(z) is of unbounded index in direction **b**.

Using calculated derivatives it can be proven that function  $F(z_1, z_2)$  is of bounded L-index in direction  $\mathbf{b}$  with  $L(z_1, z_2) = 2|-z_1 + z_2| + 1$  and  $N_{\mathbf{b}}(F, L) = 0$ .

Now we consider another function

$$F(z) = (1 + \langle z, d \rangle) \prod_{j=1}^{\infty} (1 + \langle z, c \rangle \cdot 2^{-j})^j, \ c \neq d.$$

The multiplicity of zeros for function F(z) increases to infinity. Below in this paper, we will state Theorem 12. By that theorem, unbounded multiplicity of zeros means that F(z) is of unbounded L-index in any direction  $\mathbf{b}$  ( $\langle \mathbf{b}, c \rangle \neq 0$ ) and for any positive continuous function L.

We select  $\mathbf{b} \in \mathbb{C}^n$  that  $\langle \mathbf{b}, d \rangle = 0$ . Let  $H = \{z \in \mathbb{C}^n : \langle z, d \rangle = -1\}$ . But for  $z^0 \in H$  we have

$$F(z^{0} + t\mathbf{b}) = (1 + \langle z^{0}, d \rangle + t\langle \mathbf{b}, d \rangle) \prod_{j=1}^{\infty} (1 + \langle z^{0}, c \rangle 2^{-j} + t\langle \mathbf{b}, c \rangle 2^{-j})^{j} \equiv 0.$$

Thus,  $F(z^0 + t\mathbf{b})$  is of bounded index as a function of variable t.

**Theorem 4.** Let  $(r_p)$  be a positive sequence such that  $r_p \to 1$  as  $p \to \infty$ ,  $D_p = \{z \in \mathbb{C}^n : |z| = r_p\}$ ,  $A_p$  be an everywhere dense set in  $D_p$  (i.e.  $\overline{A}_p = D_p$ ) and  $A = \bigcup_{p=1}^{\infty} A_p$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exists number M > 0 that for all  $z^0 \in A$  function  $g_{z^0}(t) = F(z^0 + t\mathbf{b})$  is of bounded  $l_{z^0}$ -index  $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ , as a function of variable  $t \in S_{z^0}$ , where  $l_{z^0}(t) \equiv L(z^0 + t\mathbf{b})$ . And  $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$ .

*Proof.* Theorem 1 implies the necessity of this theorem.

Sufficiency. It is easy to prove  $\{z + t\mathbf{b} : t \in S_z, z \in A\} = \mathbb{B}_n$ . Further, we repeat considerations with proof of sufficiency in Theorem 3 and obtain a needed conclusion.

#### 3 Auxiliary class $Q_{\mathbf{h}}^n$

The positivity and continuity of function L and condition (2) are not enough to explore the behaviour of entire function of bounded L-index in direction. Below we impose the extra condition that function L does not vary as soon.

For  $\eta \in [0, \beta]$ ,  $z \in \mathbb{B}_n$ ,  $t_0 \in S_z$  such that  $z + t_0 \mathbf{b} \in \mathbb{B}_n$  we define

$$\lambda_1^{\mathbf{b}}(z, t_0, \eta, L) = \inf \left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z + t_0\mathbf{b})} \right\},\,$$

 $\lambda_1^{\mathbf{b}}(z, \eta, L) = \inf\{\lambda_1^{\mathbf{b}}(z, t_0, \eta, L) : t_0 \in S_z\}, \ \lambda_1^{\mathbf{b}}(\eta, L) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}_n\}, \text{ and } t \in \mathbb{B}_n\}$ 

$$\lambda_2^{\mathbf{b}}(z, t_0, \eta, L) = \sup \left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z + t_0\mathbf{b})} \right\},$$

 $\lambda_2^{\mathbf{b}}(z, \eta, L) = \sup\{\lambda_2^{\mathbf{b}}(z, t_0, \eta, L) : t_0 \in S_z\}, \lambda_2^{\mathbf{b}}(\eta, L) = \sup\{\lambda_2^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}_n\}.$ 

If it will not cause misunderstandings, then  $\lambda_1^{\mathbf{b}}(z, t_0, \eta) \equiv \lambda_1^{\mathbf{b}}(z, t_0, \eta, L), \ \lambda_2^{\mathbf{b}}(z, t_0, \eta) \equiv \lambda_2^{\mathbf{b}}(z, t_0, \eta, L), \ \lambda_1^{\mathbf{b}}(z, \eta) \equiv \lambda_1^{\mathbf{b}}(z, \eta, L), \ \lambda_2^{\mathbf{b}}(z, \eta, L), \ \lambda_1^{\mathbf{b}}(\eta, L), \ \lambda_2^{\mathbf{b}}(\eta, L), \ \lambda_2^{\mathbf{b}}(\eta, L).$ 

By  $Q_{\mathbf{b},\beta}(\mathbb{B}_n)$  we denote the class of all functions L for which the following condition holds for any  $\eta \in [0,\beta]$   $0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty$ . Let  $\mathbb{D} \equiv \mathbb{B}^1$ ,  $Q_{\beta}(\mathbb{D}) \equiv Q_{1,\beta}(\mathbb{D})$ .

The following lemma suggests possible approach to compose function with  $Q_{\mathbf{b}}^n$ .

**Lemma 4.** Let  $L: \overline{\mathbb{B}}_n \to \mathbb{R}_+$  be a continuous function,  $m = \min\{L(z) : z \in \overline{\mathbb{B}}_n\}$ . Then  $\widetilde{L}(z) = \frac{\beta|\mathbf{b}|}{m} \cdot \frac{L(z)}{(1-|z|)^{\alpha}} \in Q^n_{\mathbf{b}}(\mathbb{B}_n)$  for every  $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ ,  $\alpha \geq 1$ .

*Proof.* Using definition  $Q_{\mathbf{b}}^n$  we have  $\forall z \in \mathbb{B}_n \ \forall t_0 \in S_z$ 

$$\lambda_{1}^{\mathbf{b}}(z, t_{0}, \eta, \widetilde{L}) =$$

$$= \inf \left\{ \frac{L(z+t\mathbf{b})}{(1-|z+t\mathbf{b}|)^{\alpha}} \cdot \frac{(1-|z+t_{0}\mathbf{b}|)^{\alpha}}{L(z+t_{0}\mathbf{b})} : |t-t_{0}| \leq \frac{\eta m(1-|z+t_{0}\mathbf{b}|)^{\alpha}}{\beta |b| L(z+t_{0}\mathbf{b})} \right\} \geq$$

$$\geq \inf \left\{ \frac{L(z+t\mathbf{b})}{L(z+t_{0}\mathbf{b})} : |t-t_{0}| \leq \frac{\eta m(1-|z+t_{0}\mathbf{b}|)^{\alpha}}{\beta |b| L(z+t_{0}\mathbf{b})} \right\} \times$$

$$\inf \left\{ \left( \frac{1-|z+t_{0}\mathbf{b}|}{1-|z+t\mathbf{b}|} \right)^{\alpha} : |t-t_{0}| \leq \frac{\eta m(1-|z+t_{0}\mathbf{b}|)^{\alpha}}{\beta |b| L(z+t_{0}\mathbf{b})} \right\}$$

Since Remark 1 the first infimum is not less than some constant K > 0 which is independent from z and  $t_0$ . Besides, we have  $\forall z \in \mathbb{B}_n$  and  $\forall t \in S_z \frac{m}{L(z+t_0\mathbf{b})} \leq 1$ . Thus, for the second infimum the following estimates are valid

$$\inf \left\{ \left( \frac{1 - |z + t_0 \mathbf{b}|}{1 - |z + t \mathbf{b}|} \right)^{\alpha} : |t - t_0| \le \frac{\eta m (1 - |z + t_0 \mathbf{b}|)^{\alpha}}{\beta |b| L(z + t_0 \mathbf{b})} \right\} \ge$$

$$\ge \inf \left\{ \left( \frac{1 - |z + t_0 \mathbf{b}|}{1 - |z + t \mathbf{b}|} \right)^{\alpha} : |t - t_0| \le \frac{\eta (1 - |z + t_0 \mathbf{b}|)^{\alpha}}{\beta |b|} \right\} = \left( \frac{1 - |z + t_0 \mathbf{b}|}{1 - |z + t^* \mathbf{b}|} \right)^{\alpha}.$$

where  $|t^* - t_0| \leq \frac{\eta(1-|z+t_0\mathbf{b}|)}{\beta|b|}$ . Now we find a lower estimate for this fraction

$$\frac{1 - |z + t_0 \mathbf{b}|}{1 - |z + t^* \mathbf{b}|} \ge \frac{1 - |z + t_0 \mathbf{b}|}{1 - ||z + t_0 \mathbf{b}| - |(t^* - t_0)\mathbf{b}||} \ge \frac{1 - |z + t_0 \mathbf{b}|}{1 - ||z + t_0 \mathbf{b}| - \frac{\eta(1 - |z + t_0 \mathbf{b}|)}{\beta}|}$$

Denoting  $u=|z+t_0\mathbf{b}|\in[0;1),\ \gamma=\frac{\eta}{\beta}\in[0,1],$  we consider a function of one real variable  $s(u)=\frac{1-u}{1-|u-\alpha(1-u)|}=\frac{1-u}{1-|(1+\gamma)u-\gamma|}.$  For  $u\in[0,\frac{\gamma}{\gamma+1}]$  the function s(u) strictly decreases and for  $t\in[\frac{\gamma}{1+\gamma};1)$  the function  $s(u)\equiv\frac{1}{1+\gamma}.$  In fact, we proved that

$$\lambda_1^{\mathbf{b}}(z, t_0, \eta, \widetilde{L}) \ge K \cdot \frac{1}{1 + \frac{\eta}{\beta}} > 0.$$

Hence, we have  $\lambda_1^{\mathbf{b}}(\eta, \widetilde{L}) > 0$ . By analogy it can be proved that  $\lambda_2^{\mathbf{b}}(\eta, \widetilde{L}) < \infty$ .

We often use the following properties  $Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ .

**Lemma 5.** 1. If  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$  then for every  $\theta \in \mathbb{C} \setminus \{0\}$   $L \in Q_{\theta\mathbf{b},\beta/|\theta|}(\mathbb{B}_n)$  and  $|\theta|L \in Q_{\theta\mathbf{b},\beta}(\mathbb{B}_n)$ 

2. If 
$$L \in Q_{\mathbf{b}_1,\beta}(\mathbb{B}_n) \cap Q_{\mathbf{b}_2,\beta}(\mathbb{B}_n)$$
 and for all  $z \in \mathbb{B}_n$   $L(z) > \frac{\beta \max\{|\mathbf{b}_1|,|\mathbf{b}_2|,|\mathbf{b}_1+\mathbf{b}_2|\}}{1-|z|}$  then  $\min\{\lambda_2^{\mathbf{b}_1}(\beta,L),\lambda_2^{\mathbf{b}_2}(\beta,L)\}L \in Q_{\mathbf{b}_1+\mathbf{b}_2,\beta}(\mathbb{B}_n).$ 

*Proof.* 1. First, we prove that  $(\forall \theta \in \mathbb{C} \setminus \{0\}) : L \in Q_{\theta \mathbf{b},\beta}(\mathbb{B}_n)$ . Indeed, we have by definition

$$\lambda_1^{\theta \mathbf{b}}(z, t_0, \eta, L) = \inf \left\{ \frac{L(z + t\theta \mathbf{b})}{L(z + t_0\theta \mathbf{b})} : |t - t_0| \le \frac{\eta}{L(z + t_0\theta \mathbf{b})} \right\} =$$

$$= \inf \left\{ \frac{L(z + (t\theta)\mathbf{b})}{L(z + (t_0\theta)\mathbf{b})} : |\theta t - \theta t_0| \le \frac{|\theta|\eta}{L(z + (t_0\theta)\mathbf{b})} \right\} = \lambda_1^{\mathbf{b}}(z, \theta t_0, |\theta|\eta, L).$$

Therefore, we get

$$\lambda_{1}^{\theta \mathbf{b}}(\eta, L) = \inf\{\lambda_{1}^{\theta \mathbf{b}}(z, \eta, L) : z \in \mathbb{B}_{n}\} = \inf\{\inf\{\lambda_{1}^{\theta \mathbf{b}}(z, t_{0}, \eta, L) : t_{0} \in S_{z}\} : z \in \mathbb{B}_{n}\} = \inf\{\inf\{\lambda_{1}^{\mathbf{b}}(z, \theta t_{0}, |\theta|\eta, L) : \theta t_{0} \in S_{z}\} : z \in \mathbb{B}_{n}\} = \inf\{\lambda_{1}^{\mathbf{b}}(z, |\theta|\eta, L) : z \in \mathbb{B}_{n}\} = \lambda_{1}^{\mathbf{b}}(|\theta|\eta, L) > 0,$$

because  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ . Similarly, we prove that  $\lambda_2^{\theta \mathbf{b}}(\eta, L) = \lambda_2^{\mathbf{b}}(|\theta|\eta, L) < +\infty$ . But  $|\theta|\eta \in [0, \beta]$ . So  $\eta \in [0, \beta/|\theta|]$ . Thus,  $L \in Q_{\theta \mathbf{b}, \beta/|\theta|}(\mathbb{B}_n)$ .

Let  $L^* = |\theta| \cdot L$ . Using definition of  $\lambda_1^{\mathbf{b}}(z, t_0, \eta, L^*)$  we have

$$\lambda_{1}^{\theta \mathbf{b}}(z, t_{0}, \eta, L^{*}) = \inf \left\{ \frac{L^{*}(z + t\theta \mathbf{b})}{L^{*}(z + t_{0}\theta \mathbf{b})} : |t - t_{0}| \leq \frac{\eta}{L^{*}(z + t_{0}\theta \mathbf{b})} \right\} =$$

$$= \inf \left\{ \frac{|\theta|L(z + t\theta \mathbf{b})}{|\theta|L(z + t_{0}\theta \mathbf{b})} : |t - t_{0}| \leq \frac{\eta}{|\theta|L(z + t_{0}\theta \mathbf{b})} \right\} = \inf \left\{ \frac{L(z + (t\theta)\mathbf{b})}{L(z + (t_{0}\theta)\mathbf{b})} : |\theta t - \theta t_{0}| \leq \frac{\eta}{L(z + (t_{0}\theta)\mathbf{b})} \right\} = \lambda_{1}^{\mathbf{b}}(z, \theta t_{0}, \eta, L).$$

Therefore, we obtain

$$\lambda_{1}^{\theta \mathbf{b}}(\eta, L^{*}) = \inf\{\lambda_{1}^{\theta \mathbf{b}}(z, \eta, L^{*}) : z \in \mathbb{B}_{n}\} =$$

$$= \inf\{\inf\{\lambda_{1}^{\theta \mathbf{b}}(z, t_{0}, \eta, L^{*}) : \theta t_{0} \in S_{z}\} : z \in \mathbb{B}_{n}\} =$$

$$= \inf\{\inf\{\lambda_{1}^{\mathbf{b}}(z, \theta t_{0}, \eta, L) : \theta t_{0} \in S_{z}\} : z \in \mathbb{B}_{n}\} =$$

$$= \inf\{\lambda_{1}^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}_{n}\} = \lambda_{1}^{\mathbf{b}}(\eta, L) > 0,$$

because  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ . Similarly, we prove that  $\lambda_2^{\theta \mathbf{b}}(\eta, L^*) = \lambda_2^{\mathbf{b}}(\eta, L) < +\infty$ . Thus,  $L^* = |\theta| \cdot L \in Q_{\theta \mathbf{b},\beta}(\mathbb{B}_n)$ .

2. It remains to prove a second part.

If  $z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2) \in \mathbb{B}_n$  and  $|t - t_0| \leq \frac{\eta}{L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))}$  then  $z^0 + t\mathbf{b}_1 + t_0\mathbf{b}_2 \in \mathbb{B}_n$  and  $z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2 \in \mathbb{B}_n$ . Indeed, we have

$$|z^{0} + t\mathbf{b}_{1} + t_{0}\mathbf{b}_{2}| \leq |z^{0} + t_{0}\mathbf{b}_{1} + t_{0}\mathbf{b}_{2}| + |t - t_{0}| \cdot |\mathbf{b}_{1}| \leq |z^{0} + t_{0}\mathbf{b}_{1} + t_{0}\mathbf{b}_{2}| + \frac{\eta|\mathbf{b}_{1}|}{L(z^{0} + t_{0}(\mathbf{b}_{1} + \mathbf{b}_{2}))} < |z^{0} + t_{0}\mathbf{b}_{1} + t_{0}\mathbf{b}_{2}| + \frac{\beta|\mathbf{b}_{1}|}{\frac{\beta\max\{|\mathbf{b}_{1}|,|\mathbf{b}_{2}|,|\mathbf{b}_{1} + \mathbf{b}_{2}|\}}{1 - |z^{0} + t_{0}\mathbf{b}_{1} + t_{0}\mathbf{b}_{2}|}} \leq 1.$$

Thus,  $z^0 + t\mathbf{b}_1 + t_0\mathbf{b}_2 \in \mathbb{B}_n$ .

Denote  $L^*(z) = \min\{\lambda_2^{\mathbf{b}_1}(\beta, L), \lambda_2^{\mathbf{b}_2}(\beta, L)\} \cdot L(z)$ . Assume that

$$\min\{\lambda_2^{\mathbf{b}_1}(\beta, L), \lambda_2^{\mathbf{b}_2}(\beta, L)\} = \lambda_2^{\mathbf{b}_2}(\beta, L).$$

Using definitions of  $\lambda_1^{\mathbf{b}}(\eta, L)$ ,  $\lambda_2^{\mathbf{b}}(\eta, L)$  and  $Q_{\mathbf{b},\beta}(\mathbb{B}_n)$  we obtain that

$$\inf \left\{ \frac{L^*(z^0 + t(\mathbf{b}_1 + \mathbf{b}_2))}{L^*(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} : |t - t_0| \le \frac{\eta}{L^*(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \ge$$

$$\ge \inf \left\{ \frac{L^*(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{L^*(z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{L^*(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \times$$

$$\times \inf \left\{ \frac{L^*(z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2)}{L^*(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} : |t - t_0| \le \frac{\eta}{L^*(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} =$$

$$= \inf \left\{ \frac{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{\lambda_2^{\mathbf{b}_1}(\beta, L)L(z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \times$$

$$\times \inf \left\{ \frac{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + t\mathbf{b}_2))}{\lambda_2^{\mathbf{b}_1}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + t\mathbf{b}_2))} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} =$$

$$= \inf \left\{ \frac{L(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{L(z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \times$$

$$\times \inf \left\{ \frac{L(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{L(z^0 + t_0(\mathbf{b}_1 + t\mathbf{b}_2))} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \ge$$

$$\ge \inf \left\{ \frac{L(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{L(z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \times$$

$$\times \inf \left\{ \frac{L(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{L(z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \ge$$

$$\ge \inf \left\{ \frac{L(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{L(z^0 + t_0\mathbf{b}_1 + t\mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \ge$$

$$\times \inf \left\{ \frac{L(z^0 + t\mathbf{b}_1 + t\mathbf{b}_2)}{L(z^0 + t_0(\mathbf{b}_1 + t\mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta, L)L(z^0 + t_0(\mathbf{b}_1 + \mathbf{b}_2))} \right\} \ge$$

$$\geq \inf \left\{ \frac{L(z^{0} + t\mathbf{b}_{1} + t\mathbf{b}_{2})}{L(z^{0} + t_{0}\mathbf{b}_{1} + t\mathbf{b}_{2})} : |t - t_{0}| \leq \frac{\eta}{\lambda_{2}^{\mathbf{b}_{2}}(\beta, L)L(z^{0} + t_{0}(\mathbf{b}_{1} + \mathbf{b}_{2}))} \right\} \times \\ \times \lambda_{1}^{\mathbf{b}_{2}}(z^{0} + t_{0}\mathbf{b}_{1}, t_{0}, \eta, L) \geq \lambda_{1}^{\mathbf{b}_{2}}(\eta, L) \frac{L(z^{0} + t\hat{\mathbf{b}}_{1} + t\hat{\mathbf{b}}_{2})}{L(z^{0} + t_{0}\mathbf{b}_{1} + t\hat{\mathbf{b}}_{2})}$$
(8)

where  $\hat{t}$  is a point at which infimum is attained

$$\frac{L(z^0+\hat{t}\mathbf{b}_1+\hat{t}\mathbf{b}_2)}{L(z^0+t_0\mathbf{b}_1+\hat{t}\mathbf{b}_2)} = \inf \left\{ \frac{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{L(z^0+t_0\mathbf{b}_1+t\mathbf{b}_2)} : |t-t_0| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0+t_0(\mathbf{b}_1+\mathbf{b}_2))} \right\}.$$

But  $L \in Q_{\mathbf{b}_2,\beta}(\mathbb{B}_n)$ , then for all  $\eta \in [0,\beta]$ 

$$\sup \left\{ \frac{L(z^0 + t_0 \mathbf{b}_1 + t \mathbf{b}_2)}{L(z^0 + t_0 \mathbf{b}_1 + t_0 \mathbf{b}_2)} : |t - t_0| \le \frac{\eta}{L(z^0 + t_0 (\mathbf{b}_1 + \mathbf{b}_2))} \right\} \le \lambda_2^{\mathbf{b}_2}(\eta, L) < \infty.$$

Hence,  $L(z^0+t_0\mathbf{b}_1+t\mathbf{b}_2) \leq \lambda_2^{\mathbf{b}_2}(\eta, L) \cdot L(z^0+t_0\mathbf{b}_1+t_0\mathbf{b}_2)$ , i.e. for  $t=\hat{t}$  we have  $L(z^0+t_0\mathbf{b}_1+t_0\mathbf{b}_2) \geq \frac{L(z^0+t_0\mathbf{b}_1+\hat{t}\mathbf{b}_2)}{\lambda_2^{\mathbf{b}_2}(\eta, L)}$ . Using a proved inequality and (8), we obtain

$$\inf \left\{ \frac{L^{*}(z^{0} + t(\mathbf{b}_{1} + \mathbf{b}_{2}))}{L^{*}(z^{0} + t_{0}(\mathbf{b}_{1} + \mathbf{b}_{2}))} : |t - t_{0}| \leq \frac{\eta}{L^{*}(z^{0} + t_{0}(\mathbf{b}_{1} + \mathbf{b}_{2}))} \right\} \geq$$

$$\geq \lambda_{1}^{\mathbf{b}_{2}}(\eta, L) \cdot \inf \left\{ \frac{L(z^{0} + t\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})}{L(z^{0} + t_{0}\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})} : |t - t_{0}| \leq \frac{\eta}{\lambda_{2}^{\mathbf{b}_{2}}(\beta, L)L(z^{0} + t_{0}(\mathbf{b}_{1} + \mathbf{b}_{2}))} \right\} \geq$$

$$\geq \lambda_{1}^{\mathbf{b}_{2}}(\eta, L) \cdot \inf \left\{ \frac{L(z^{0} + t\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})}{L(z^{0} + t_{0}\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})} : |t - t_{0}| \leq \frac{\eta\lambda_{2}^{\mathbf{b}_{2}}(\beta, L)L(z^{0} + t_{0}\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})}{\lambda_{2}^{\mathbf{b}_{2}}(\beta, L)L(z^{0} + t_{0}\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})} \right\} \geq$$

$$\geq \lambda_{1}^{\mathbf{b}_{2}}(\eta, L) \cdot \inf \left\{ \frac{L(z^{0} + t\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})}{L(z^{0} + t_{0}\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})} : |t - t_{0}| \leq \frac{\eta}{L(z^{0} + t_{0}\mathbf{b}_{1} + \hat{t}\mathbf{b}_{2})} \right\} =$$

$$= \lambda_{1}^{\mathbf{b}_{2}}(\eta, L)\lambda_{1}^{\mathbf{b}_{1}}(z^{0} + \hat{t}\mathbf{b}_{2}, t_{0}, \eta, L) \geq \lambda_{1}^{\mathbf{b}_{2}}(\eta, L)\lambda_{1}^{\mathbf{b}_{1}}(\eta, L).$$

Therefore,  $\lambda_1^{\mathbf{b}_1+\mathbf{b}_2}(\eta, L^*) \geq \lambda_1^{\mathbf{b}_2}(\eta, L)\lambda_1^{\mathbf{b}_1}(\eta, L) > 0$ . By analogy, we can prove that for all  $\eta \in [0, \beta]$   $\lambda_2^{\mathbf{b}_1+\mathbf{b}_2}(\eta, L^*) < +\infty$ . Thus,  $L^* \in Q_{\mathbf{b}_1+\mathbf{b}_2,\beta}(\mathbb{B}_n)$ .

### 4 Criteria of L-index boundedness in direction, related to the behaviour of the function F.

The following theorem is an analogue of Theorem 2 from [4].

**Theorem 5.** Let  $\beta > 1$  and  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if for every  $\eta$ ,  $0 < \eta \leq \beta$ , there exist  $n_0 = n_0(\eta) \in \mathbb{Z}_+$  and  $P_1 = P_1(\eta) \geq 1$  that for each  $z \in \mathbb{B}_n$  and each  $t_0 \in S_z$  there exists  $k_0 = k_0(t_0, z) \in \mathbb{Z}_+$ , with  $0 \leq k_0 \leq n_0$ , and the following inequality holds

$$\max\left\{ \left| \frac{\partial^{k_0} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})} \right\} \le P_1 \left| \frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}} \right|. \tag{9}$$

*Proof.* Necessity. Let F be of bounded L-index in direction  $\mathbf{b}$  and  $N_{\mathbf{b}}(F;L) \equiv N < +\infty$ . We denote

$$q(\eta) = [2\eta(N+1)(\lambda_2^{\mathbf{b}}(\eta))^{N+1}(\lambda_1^{\mathbf{b}}(\eta))^{-N}] + 1,$$

where [a] is an entire part of number  $a \in \mathbb{R}$ . For  $z \in \mathbb{B}_n$ ,  $t_0 \in S_z$  and  $p \in \{0, 1, \dots, q(\eta)\}$  we put

$$R_p^{\mathbf{b}}(z, t_0, \eta) = \max \left\{ \frac{1}{k! L^k(z + t\mathbf{b})} \left| \frac{\partial^k F(z + t\mathbf{b})}{\partial \mathbf{b}^k} \right| : |t - t_0| \le \frac{p\eta}{q(\eta) L(z + t_0\mathbf{b})}, 0 \le k \le N \right\}.$$

and

$$\widetilde{R}_{p}^{\mathbf{b}}(z, t_{0}, \eta) = \max \left\{ \frac{1}{k! L^{k}(z + t_{0}\mathbf{b})} \left| \frac{\partial^{k} F(z + t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : |t - t_{0}| \le \frac{p\eta}{q(\eta) L(z + t_{0}\mathbf{b})}, 0 \le k \le N \right\}.$$
But  $|t - t_{0}| \le \frac{p\eta}{q(\eta) L(z + t_{0}\mathbf{b})} \le \frac{\eta}{L(z + t_{0}\mathbf{b})} \le \frac{\beta}{L(z + t_{0}\mathbf{b})}, \text{ then}$ 

$$\lambda_{1}^{\mathbf{b}} \left( z, t_{0}, \frac{p\eta}{q(\eta)} \right) \ge \lambda_{1}^{\mathbf{b}}(z, t_{0}, \eta) \ge \lambda_{1}^{\mathbf{b}}(\eta), \ \lambda_{2}^{\mathbf{b}} \left( z, t_{0}, \frac{p\eta}{q(\eta)} \right) \le \lambda_{2}^{\mathbf{b}}(z, t_{0}, \eta) \le \lambda_{2}^{\mathbf{b}}(\eta).$$

Clearly, these quantities  $R_p^{\mathbf{b}}(z, t_0, \eta)$ ,  $\widetilde{R}_p^{\mathbf{b}}(z, t_0, \eta)$  are defined. Besides,

$$R_{p}^{\mathbf{b}}(z, t_{0}, \eta) = \max \left\{ \frac{1}{k!L^{k}(z + t_{0}\mathbf{b})} \left| \frac{\partial^{k}F(z + t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| \left( \frac{L(z + t_{0}\mathbf{b})}{L(z + t\mathbf{b})} \right)^{k} : \right.$$

$$\left. | t - t_{0}| \leq \frac{p\eta}{q(\eta)L(z + t_{0}\mathbf{b})}, 0 \leq k \leq N \right\} \leq$$

$$\leq \max \left\{ \frac{1}{k!L^{k}(z + t_{0}\mathbf{b})} \left| \frac{\partial^{k}F(z + t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| \left( \frac{1}{\lambda_{1}^{\mathbf{b}}(z, t_{0}, \frac{p\eta}{q(\eta)})} \right)^{k} : \right.$$

$$\left. | t - t_{0}| \leq \frac{p\eta}{q(\eta)L(z + t_{0}\mathbf{b})}, 0 \leq k \leq N \right\} \leq$$

$$\leq \max \left\{ \frac{1}{k!L^{k}(z + t_{0}\mathbf{b})} \left| \frac{\partial^{k}F(z + t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| \left( \frac{1}{\lambda_{1}^{\mathbf{b}}(\eta)} \right)^{k} : \right.$$

$$\left. | t - t_{0}| \leq \frac{p\eta}{q(\eta)L(z + t_{0}\mathbf{b})}, 0 \leq k \leq N \right\} \leq$$

$$\leq \left( \frac{1}{\lambda_{1}^{\mathbf{b}}(\eta)} \right)^{N} \max \left\{ \frac{1}{k!L^{k}(z + t_{0}\mathbf{b})} \left| \frac{\partial^{k}F(z + t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : \right.$$

$$\left. | t - t_{0}| \leq \frac{p\eta}{q(\eta)L(z + t_{0}\mathbf{b})}, 0 \leq k \leq N \right\} = \widetilde{R}_{p}^{\mathbf{b}}(z, t_{0}, \eta)(\lambda_{1}^{\mathbf{b}}(\eta))^{-N}$$

$$\left. | t - t_{0}| \leq \frac{p\eta}{q(\eta)L(z + t_{0}\mathbf{b})}, 0 \leq k \leq N \right\} = \widetilde{R}_{p}^{\mathbf{b}}(z, t_{0}, \eta)(\lambda_{1}^{\mathbf{b}}(\eta))^{-N}$$

$$(10)$$

and

$$\widetilde{R}_{p}^{\mathbf{b}}(z, t_{0}, \eta) = \max \left\{ \frac{1}{k! L^{k}(z + t\mathbf{b})} \left| \frac{\partial^{k} F(z + t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| \left( \frac{L(z + t\mathbf{b})}{L(z + t_{0}\mathbf{b})} \right)^{k} : |t - t_{0}| \le \frac{p\eta}{q(\eta)L(z + t_{0}\mathbf{b})}, 0 \le k \le N \right\} \le$$

$$\leq \max \left\{ \frac{1}{k!L^{k}(z+t\mathbf{b})} \left| \frac{\partial^{k}F(z+t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| \left( \lambda_{2}^{\mathbf{b}} \left( z, t_{0}, \frac{p\eta}{q(\eta)} \right) \right)^{k} : \right. \\
\left. \left. \left| t - t_{0} \right| \leq \frac{p\eta}{q(\eta)L(z+t_{0}\mathbf{b})}, 0 \leq k \leq N \right\} \leq \\
\leq \max \left\{ \frac{\left( \lambda_{2}^{\mathbf{b}}(\eta) \right)^{k}}{k!L^{k}(z+t\mathbf{b})} \left| \frac{\partial^{k}F(z+t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : \left| t - t_{0} \right| \leq \frac{p\eta}{q(\eta)L(z+t_{0}\mathbf{b})}, \\
0 \leq k \leq N \right\} \leq \left( \lambda_{2}^{\mathbf{b}}(\eta) \right)^{N} \max \left\{ \frac{1}{k!L^{k}(z+t\mathbf{b})} \left| \frac{\partial^{k}F(z+t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : \\
\left| t - t_{0} \right| \leq \frac{p\eta}{q(\eta)L(z+t_{0}\mathbf{b})}, 0 \leq k \leq N \right\} = R_{p}^{\mathbf{b}}(z, t_{0}, \eta)(\lambda_{2}^{\mathbf{b}}(\eta))^{N}. \tag{11}$$

Let  $k_p^z \in \mathbb{Z}$ ,  $0 \le k_p^z \le N$ , and  $t_p^z \in \mathbb{C}$ ,  $|t_p^z - t_0| \le \frac{p\eta}{q(\eta)L(z + t_0\mathbf{b})}$ , be such that

$$\frac{1}{k_p^z! L^{k_p^z}(z + t_0 \mathbf{b})} \left| \frac{\partial^{k_p^z} F(z + t_p^z \mathbf{b})}{\partial \mathbf{b}^{k_p^z}} \right| = \widetilde{R}_p^{\mathbf{b}}(z, t_0, \eta).$$
 (12)

For every given  $z \in \mathbb{B}_n$  a function  $F(z + t\mathbf{b})$  and its directional derivative are analytic. By the maximum modulus principle an equality (12) holds for such  $t_p^z$ , that

$$|t_p^z - t_0| = \frac{p\eta}{q(\eta)L(z + t_0\mathbf{b})}.$$

We put  $\widetilde{t_p^z} = t_0 + \frac{p-1}{p}(t_p^z - t_0)$ . Then

$$|\widetilde{t}_p^z - t_0| = \frac{(p-1)\eta}{q(\eta)L(z+t_0\mathbf{b})}$$
(13)

and

$$|\tilde{t}_p^z - t_p^z| = \frac{|t_p^z - t_0|}{p} = \frac{\eta}{q(\eta)L(z + t_0 \mathbf{b})}.$$
 (14)

In view of (13) and the definition of  $\widetilde{R}_{p-1}^{\mathbf{b}}(z,t_0,\eta)$ , we obtain that

$$\widetilde{R}_{p-1}^{\mathbf{b}}(z,t_0,\eta) \ge \frac{1}{k_p^z! L^{k_p^z}(z+t_0\mathbf{b})} \left| \frac{\partial^{k_p^z} F(z+\widetilde{t_p^z}\mathbf{b})}{\partial \mathbf{b}^{k_p^z}} \right|.$$

Therefore, this inequality holds

$$0 \leq \widetilde{R}_{p}^{\mathbf{b}}(z, t_{0}, \eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z, t_{0}, \eta) \leq \frac{\left| \frac{\partial^{k_{p}^{z}} F(z + t_{p}^{z} \mathbf{b})}{\partial \mathbf{b}^{k_{p}^{z}}} \right| - \left| \frac{\partial^{k_{p}^{z}} F(z + \widetilde{t_{p}^{z}} \mathbf{b})}{\partial \mathbf{b}^{k_{p}^{z}}} \right|}{k_{p}^{z}! L^{k_{p}^{z}}(z + t_{0} \mathbf{b})} = \frac{1}{k_{p}^{z}! L^{k_{p}^{z}}(z + t_{0} \mathbf{b})} \int_{0}^{1} \frac{d}{ds} \left| \frac{\partial^{k_{p}^{z}} F(z + (\widetilde{t_{p}^{z}} + s(t_{p}^{z} - \widetilde{t_{p}^{z}})) \mathbf{b})}{\partial \mathbf{b}^{k_{p}^{z}}} \right| ds.$$

$$(15)$$

For every analytic complex-valued function of real variable  $\varphi(s)$ ,  $s \in \mathbb{R}$ , the inequality  $\frac{d}{ds}|\varphi(s)| \le \left|\frac{d}{ds}\varphi(s)\right|$  holds except the points where  $\varphi(s) = 0$ . Applying this inequality to (15) and using a mean value theorem, we have

$$\begin{split} \widetilde{R}_{p}^{\mathbf{b}}(z,t_{0},\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,t_{0},\eta) &\leq \frac{|t_{p}^{z} - \widetilde{t}_{p}^{z}|}{k_{p}^{z}!L^{k_{p}^{z}}(z+t_{0}\mathbf{b})} \int_{0}^{1} \left| \frac{\partial^{k_{p}^{z}+1}F(z+(\widetilde{t}_{p}^{z}+s(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})}{\partial \mathbf{b}^{k_{p}^{z}+1}} \right| ds = \\ &= \frac{|t_{p}^{z} - \widetilde{t}_{p}^{z}|}{k_{n}^{p}!L^{k_{p}^{z}}(z+t_{0}\mathbf{b})} \left| \frac{\partial^{k_{p}^{z}+1}F(z+(\widetilde{t}_{p}^{z}+s^{*}(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})}{\partial \mathbf{b}^{k_{p}^{z}+1}} \right| = \\ &= \frac{1}{(k_{p}^{z}+1)!L^{k_{p}^{z}+1}(z+t_{0}\mathbf{b})} \left| \frac{\partial^{k_{p}^{z}+1}F(z+(\widetilde{t}_{p}^{z}+s^{*}(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})}{\partial \mathbf{b}^{k_{p}^{z}+1}} \right| \cdot L(z+t_{0}\mathbf{b})(k_{p}^{z}+1)|t_{p}^{z}-\widetilde{t}_{p}^{z}|, \end{split}$$

where  $s^* \in [0, 1]$ .

The point  $\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z)$  lies into the set

$$\left\{ t \in \mathbb{C} : |t - t_0| \le \frac{p\eta}{q(\eta)L(z + t_0\mathbf{b})} \le \frac{\eta}{L(z + t_0\mathbf{b})} \right\}.$$

Using L-index boundedness in direction **b** of function F, definition  $q(\eta)$ , inequality (10) and (14), for  $k_p^z \leq N$  we have

$$\begin{split} \widetilde{R}_{p}^{\mathbf{b}}(z,t_{0},\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,t_{0},\eta) &\leq \frac{1}{(k_{n}^{z}+1)!L^{k_{n}^{z}+1}(z+(\widetilde{t}_{p}^{z}+s^{*}(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})} \times \\ &\times \left| \frac{\partial^{k_{p}^{z}+1}F(z+(\widetilde{t}_{p}^{z}+s^{*}(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})}{\partial \mathbf{b}^{k_{p}^{z}+1}} \right| \left( \frac{L(z+(\widetilde{t}_{p}^{z}+s^{*}(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})}{L(z+t_{0}\mathbf{b})} \right)^{k_{p}^{z}+1} \times \\ &\times L(z+t_{0}\mathbf{b})(k_{n}^{z}+1)|t_{p}^{z}-\widetilde{t}_{p}^{z}| \leq \eta \frac{N+1}{q(\eta)}(\lambda_{2}^{\mathbf{b}}(z,t_{0},\eta))^{N+1} \times \\ &\times \max \left\{ \frac{1}{k!L^{k}(z+(\widetilde{t}_{p}^{z}+s^{*}(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})} \left| \frac{\partial^{k}F(z+(\widetilde{t}_{p}^{z}+s^{*}(t_{p}^{z}-\widetilde{t}_{p}^{z}))\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : 0 \leq k \leq N \right\} \leq \\ &\leq \eta \frac{N+1}{q(\eta)}(\lambda_{2}^{\mathbf{b}}(\eta))^{N+1}R_{p}^{\mathbf{b}}(z,t_{0},\eta) \leq \frac{\eta(N+1)(\lambda_{2}^{\mathbf{b}}(\eta))^{N+1}(\lambda_{1}^{\mathbf{b}}(\eta))^{-N}}{[2\eta(N+1)\lambda_{2}^{\mathbf{b}}(\eta)(\lambda_{1}^{\mathbf{b}}(\eta))^{-N}]+1}\widetilde{R}_{p}^{\mathbf{b}}(z,t_{0},\eta) \leq \\ &\leq \frac{1}{2}\widetilde{R}_{p}^{\mathbf{b}}(z,t_{0},\eta) \end{split}$$

In the last inequality we used that  $2a + 1 \ge [2a + 1] = [2a] + 1 \ge 2a$  for  $a \in \mathbb{R}$ .

It follows that  $\widetilde{R}_p^{\mathbf{b}}(z, t_0, \eta) \leq 2\widetilde{R}_{p-1}^{\mathbf{b}}(z, t_0, \eta)$ . Using inequalities (10) and (11), we deduce for  $R_p^{\mathbf{b}}(z, t_0, \eta)$ 

$$R_p^{\mathbf{b}}(z, t_0, \eta) \le 2(\lambda_1^{\mathbf{b}}(\eta))^{-N} \widetilde{R}_{p-1}^{\mathbf{b}}(z, t_0, \eta) \le 2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N} R_{p-1}^{\mathbf{b}}(z, t_0, \eta).$$

Hence, we have

$$\max \left\{ \frac{1}{k!L^k(z+t\mathbf{b})} \left| \frac{\partial^k F(z+t\mathbf{b})}{\partial \mathbf{b}^k} \right| : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})}, \right.$$

$$0 \leq k \leq N \} = R_{q(\eta)}^{\mathbf{b}}(z, t_0, \eta) \leq 2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N} R_{q(\eta)-1}^{\mathbf{b}}(z, t_0, \eta) \leq$$

$$\leq (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^2 R_{q(\eta)-2}^{\mathbf{b}}(z, t_0, \eta) \leq \cdots \leq$$

$$\leq (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^{q(\eta)} R_0^{\mathbf{b}}(z, t_0, \eta) = (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^{q(\eta)} \times$$

$$\times \max \left\{ \frac{1}{k! L^k(z + t_0 \mathbf{b})} \left| \frac{\partial^k F(z + t_0 \mathbf{b})}{\partial \mathbf{b}^k} \right| : 0 \leq k \leq N \right\}.$$

$$(16)$$

Let  $k_0^z \in \mathbb{Z}$ ,  $0 \le k_0^z = k_0^z(t_0) \le N$ , and  $\widetilde{t}^z \in \mathbb{C}$ ,  $|\widetilde{t}^z - t_0| = \frac{\eta}{L(z + t_0 \mathbf{b})}$ , be defined as

$$\frac{1}{k_0^z! L^{k_0^z}(z+t_0\mathbf{b})} \left| \frac{\partial^{k_0^z} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| = \max_{0 \le k \le N} \left\{ \frac{1}{k! L^k(z+t_0\mathbf{b})} \left| \frac{\partial^k F(z+t_0\mathbf{b})}{\partial \mathbf{b}^k} \right| \right\}$$

and

$$\left| \frac{\partial^{k_0^z} F(z + \widetilde{t}^z \mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| = \max \left\{ \left| \frac{\partial^{k_0^z} F(z + t \mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| : |t - t_0| \le \frac{\eta}{L(z + t_0 \mathbf{b})} \right\}.$$

From inequality (16) it follows

$$\frac{1}{k_0^z! L^{k_0^z}(z + \tilde{t}^z \mathbf{b})} \left| \frac{\partial^{k_0^z} F(z + \tilde{t}^z \mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| \leq \\
\leq \max \left\{ \frac{1}{k_0^z! L^{k_0^z}(z + t \mathbf{b})} \left| \frac{\partial^{k_0^z} F(z + t \mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| : |t - t_0| = \frac{\eta}{L(z + t_0 \mathbf{b})} \right\} \leq \\
\leq \max \left\{ \frac{1}{k! L^k(z + t \mathbf{b})} \left| \frac{\partial^k F(z + t \mathbf{b})}{\partial \mathbf{b}^k} \right| : |t - t_0| = \frac{\eta}{L(z + t_0 \mathbf{b})}, \\
0 \leq k \leq N \right\} \leq (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^{q(\eta)} \frac{1}{k_0^z! L^{k_0^z}(z + t_0 \mathbf{b})} \cdot \left| \frac{\partial^{k_0^z} F(z + t_0 \mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right|.$$

Hence, we get

$$\max \left\{ \left| \frac{\partial^{k_0^z} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})} \right\} \le$$

$$\le (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^{q(\eta)} \left( \frac{L(z+\tilde{t}^z\mathbf{b})}{L(z+t_0\mathbf{b})} \right)^{k_0^z} \left| \frac{\partial^{k_0^z} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| \le$$

$$\le (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^{q(\eta)} (\lambda_2^{\mathbf{b}}(z,t_0,\eta))^N \left| \frac{\partial^{k_0^z} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right| \le$$

$$\le (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^{q(\eta)} (\lambda_2^{\mathbf{b}}(\eta))^N \left| \frac{\partial^{k_0^z} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0^z}} \right|.$$

We proved (9) with  $n_0 = N_{\mathbf{b}}(F, L)$  and

$$P_1(\eta) = (2(\lambda_2^{\mathbf{b}}(\eta))^N (\lambda_1^{\mathbf{b}}(\eta))^{-N})^{q(\eta)} (\lambda_2^{\mathbf{b}}(\eta))^N > 1.$$

**Sufficiency.** Suppose that for each  $\eta \in (0, \beta]$  there exist  $n_0 = n_0(\eta) \in \mathbb{Z}_+$  and  $P_1 = P_1(\eta) \ge 1$  that for every  $z \in \mathbb{B}_n$  and for every  $t_0 \in S_z$  there exists  $k_0 = k_0(t_0, z) \in \mathbb{Z}_+$ ,  $0 \le k_0 \le n_0$ , for which inequality (9) holds. But  $\eta$  is arbitrary in  $(0, \beta]$  and  $\beta > 1$  then we can pick  $\eta > 1$ . We

select  $j_0 \in \mathbb{N}$  satisfying  $P_1 \leq \eta^{j_0}$ . For given  $z \in \mathbb{B}_n$ ,  $t_0 \in S_z$ , suiting  $k_0 = k_0(t_0, z)$  and  $j \geq j_0$  by Cauchy formula for  $F(z + t\mathbf{b})$  as a function of one variable t

$$\frac{\partial^{k_0+j}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0+j}} = \frac{j!}{2\pi i} \int_{|t-t_0|=n/L(z+t_0\mathbf{b})} \frac{1}{(t-t_0)^{j+1}} \frac{\partial^{k_0}F(z+t\mathbf{b})}{\partial\mathbf{b}^{k_0}} dt.$$

Since (9) we have

$$\frac{1}{j!} \left| \frac{\partial^{k_0+j} F(z+t_0 \mathbf{b})}{\partial \mathbf{b}^{k_0+j}} \right| \le \frac{L^j(z+t_0 \mathbf{b})}{\eta^j} \max \left\{ \left| \frac{\partial^{k_0} F(z+t \mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| : |t-t_0| = \frac{\eta}{L(z+t_0 \mathbf{b})} \right\} \le P_1 \frac{L^j(z+t_0 \mathbf{b})}{\eta^j} \left| \frac{\partial^{k_0} F(z+t_0 \mathbf{b})}{\partial \mathbf{b}^{k_0}} \right|,$$

that is

$$\frac{1!}{(k_0+j)!L^{k_0+j}(z+t_0\mathbf{b})} \left| \frac{\partial^{k_0+j}F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0+j}} \right| \leq \frac{j!k_0!}{(j+k_0)!} \frac{P_1}{\eta^j} \times \frac{1}{k_0!L^{k_0}(z+t_0\mathbf{b})} \left| \frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| \leq \eta^{j_0-j} \frac{1}{k_0!L^{k_0}(z+t_0\mathbf{b})} \times \left| \frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| \leq \frac{1}{k_0!L^{k_0}(z+t_0\mathbf{b})} \left| \frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}} \right|$$

for all  $j \geq j_0$ .

In the above inequality  $k_0 \leq n_0$ ,  $n_0 = n_0(\eta)$  and  $j_0 = j_0(\eta)$  are independent of z and  $t_0$ . Since  $z \in \mathbb{B}_n$  and  $t_0 \in \mathbb{B}_z$  are arbitrary, this inequality means that function F is of bounded L-index in direction  $\mathbf{b}$  and  $N_{\mathbf{b}}(F, L) \leq n_0 + j_0$ . Theorem 5 is proved.

**Theorem 6.** Let  $\beta > 1$ ,  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ ,  $\frac{1}{\beta} < \theta_1 \le \theta_2 < +\infty$ ,  $\theta_1 L(z) \le L^*(z) \le \theta_2 L(z)$ . Analytic in  $\mathbb{B}_n$  function F(z),  $z \in \mathbb{C}^n$ , is of bounded  $L^*$ -index in direction  $\mathbf{b}$  if and only if F is of bounded L-index in direction  $\mathbf{b}$ .

Proof. Obviously, if  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$  and  $\theta_1 L(z) \leq L^*(z) \leq \theta_2 L(z)$ , then  $L^* \in Q_{\mathbf{b},\beta^*}(\mathbb{B}_n)$ ,  $\beta^* \in [\theta_1\beta;\theta_2\beta]$  and  $\beta^* > 1$ . Let  $N_{\mathbf{b}}(F,L^*) < +\infty$ . Therefore, by Theorem 5 for each  $\eta^*$ ,  $0 < \eta^* < \beta\theta_2$ , there exist  $n_0(\eta^*) \in \mathbb{Z}_+$  and  $P_1(\eta^*) \geq 1$  that for every  $z \in \mathbb{B}_n$ ,  $t_0 \in S_z$  and some  $k_0$ ,  $0 \leq k_0 \leq n_0$ , the inequality (9) is valid with  $L^*$  and  $\eta^*$  instead of L and  $\eta$ . Hence, we put  $\eta^* = \theta_2 \eta$  and obtain

$$P_{1} \left| \frac{\partial^{k_{0}} F(z+t_{0}\mathbf{b})}{\partial \mathbf{b}^{k_{0}}} \right| \ge \max \left\{ \left| \frac{\partial^{k_{0}} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_{0}}} \right| : |t-t_{0}| \le \frac{\eta^{*}}{L^{*}(z+t_{0}\mathbf{b})} \right\} \ge$$

$$\ge \max \left\{ \left| \frac{\partial^{k_{0}} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_{0}}} \right| : |t-t_{0}| \le \frac{\eta}{L(z+t_{0}\mathbf{b})} \right\}.$$

Therefore, by Theorem 5, the function F(z) is of bounded L-index in direction **b**. The converse assertion is obtained by replacing L on  $L^*$ .

**Theorem 7.** Let  $\beta > 1$ ,  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ ,  $m \in \mathbb{C}$ ,  $m \neq 0$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if F(z) is of bounded L-index in direction  $m\mathbf{b}$ .

Proof. Let F(z) be an analytic in  $\mathbb{B}_n$  function of bounded L-index in direction **b**. By Theorem 5  $(\forall \eta > 0)$   $(\exists n_0(\eta) \in \mathbb{Z}_+)$   $(\exists P_1(\eta) \geq 1)$   $(\forall z \in \mathbb{B}_n)$   $(\forall t_0 \in S_z)$   $(\exists k_0 = k_0(t_0, z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0)$ , and the following inequality is valid

$$\max\left\{ \left| \frac{\partial^{k_0} F(z+t\mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})} \right\} \le P_1 \left| \frac{\partial^{k_0} F(z+t_0\mathbf{b})}{\partial \mathbf{b}^{k_0}} \right|. \tag{17}$$

Since  $\frac{\partial^k F}{\partial (m\mathbf{b})^k} = (m)^k \frac{\partial^k F}{\partial \mathbf{b}^k}$ , the inequality (17) is equivalent to the inequality

$$\max\left\{|m|^{k_0}\left|\frac{\partial^{k_0}F(z+t\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right|:|t-t_0|\leq \frac{\eta}{L(z+t_0\mathbf{b})}\right\}\leq P_1|m|^{k_0}\left|\frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right|$$

or

$$\max \left\{ \left| \frac{\partial^{k_0} F(z + \frac{t}{m} m \mathbf{b})}{\partial (m \mathbf{b})^{k_0}} \right| : \left| \frac{t - t_0}{m} \right| \le \frac{\eta}{|m| L(z + \frac{t_0}{m} m \mathbf{b})} \right\} \le P_1 \left| \frac{\partial^{k_0} F(z + \frac{t_0}{m} m \mathbf{b})}{\partial (m \mathbf{b})^{k_0}} \right|.$$

Denoting  $t^* = \frac{t}{m}$ ,  $t_0^* = \frac{t_0}{m}$ ,  $\eta^* = \frac{\eta}{|m|}$ , we obtain

$$\max\left\{\left|\frac{\partial^{k_0}F(z+t^*m\mathbf{b})}{\partial(m\mathbf{b})^{k_0}}\right|:|t^*-t_0^*|\leq \frac{\eta^*}{L(z+t_0^*m\mathbf{b})}\right\}\leq P_1\left|\frac{\partial^{k_0}F(z+t_0\mathbf{b})}{\partial\mathbf{b}^{k_0}}\right|.$$

By Theorem 5 a function F(z) is of bounded L-index in direction **b**. Similarly, the converse assertion can be proved.

# 5 Estimate of maximum modulus on a larger circle by maximum modulus on a smaller circle and by minimum modulus.

Now we consider a behaviour of analytic in the unit ball functions of bounded L-index in direction. Using Theorem 5, we prove a criterion of L-index boundedness in direction.

**Theorem 8.** Let  $\beta > 1$ ,  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if for any  $r_1$  and any  $r_2$  with  $0 < r_1 < r_2 \le \beta$ , there exists number  $P_1 = P_1(r_1, r_2) \ge 1$  such that for each  $z^0 \in \mathbb{B}_n$  and each  $t_0 \in S_{z^0}$ 

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r_{2}}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{1} \max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r_{1}}{L(z_{0} + t_{0}\mathbf{b})} \right\}.$$
(18)

*Proof.* Necessity. Let  $N_{\mathbf{b}}(F, L) < +\infty$ . On the contrary, we assume there exists numbers  $r_1$  and  $r_2$ ,  $0 < r_1 < r_2 \le \beta$ , that for every  $P_* \ge 1$  there exist  $z^* = z^*(P_*) \in \mathbb{B}_n$  and  $t^* = t^*(P^*) \in S_{z^*}$ , the following inequality is valid

$$\max \{ |F(z^* + t\mathbf{b})| : |t - t^*| = \frac{r_2}{L(z^* + t^*\mathbf{b})} \} >$$

$$> P_* \max \{ |F(z^* + t\mathbf{b})| : |t - t^*| = \frac{r_1}{L(z^* + t^*\mathbf{b})} \}.$$

By Theorem 5 there exist  $n_0 = n_0(r_2) \in \mathbb{Z}_+$  and  $P_0 = P_0(r_2) \ge 1$  that for every  $z^* \in \mathbb{B}_n$ ,  $t^* \in S_{z^*}$  and some  $k_0 = k_0(t^*, z^*) \in \mathbb{Z}_+$ ,  $0 \le k_0 \le n_0$ , the following inequality holds

$$\max\left\{\left|\frac{\partial^{k_0} F(z^* + t\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right| : |t - t^*| = \frac{r_2}{L(z^* + t^*\mathbf{b})}\right\} \le P_0 \left|\frac{\partial^{k_0} F(z^* + t^*\mathbf{b})}{\partial \mathbf{b}^{k_0}}\right|. \tag{19}$$

We remark that for  $k_0 = 0$  the proof of necessity is obvious because (19) implies  $\max \{|F(z^* + t\mathbf{b})| : |t - t^*| = r_2/L(z^* + t^*\mathbf{b})\} \le P_0|F(z^* + t^*\mathbf{b})| \le P_0 \max\{|F(z^* + t\mathbf{b})| : |t - t^*| = r_1/L(z^* + t^*\mathbf{b})\}.$  We assume that  $k_0 > 0$ , and let

$$P_* = n_0! \left(\frac{r_2}{r_1}\right)^{n_0} \left(P_0 + \frac{r_1}{r_2 - r_1}\right) + 1.$$
 (20)

Let  $t_0 \in S_{z^*}$  be such that  $|t_0 - t^*| = r_1/L(z^* + t^*\mathbf{b})$  and

$$|F(z^* + t_0 \mathbf{b})| = \max\{|F(z^* + t\mathbf{b})| : |t - t^*| = r_1/L(z^* + t^*\mathbf{b})\} > 0,$$

but  $t_{0j} \in S_{z^*}$ ,  $|t_{0j} - t^*| = r_2/L(z^* + t^*\mathbf{b})$ , be such that

$$\left| \frac{\partial^j F(z^* + t_{0j} \mathbf{b})}{\partial \mathbf{b}^j} \right| = \max \left\{ \left| \frac{\partial^j F(z^* + t \mathbf{b})}{\partial \mathbf{b}^j} \right| : |t - t^*| = r_2 / L(z^* + t^* \mathbf{b}) \right\}, \ j \in \mathbb{Z}_+.$$

We remark that in the case  $|F(z^*+t_0\mathbf{b})|=0$  by the uniqueness theorem for all  $t \in S_{z^*}$  an equality  $F(z^*+t\mathbf{b})=0$  can be obtained. However, it contradicts an inequality (5). By Cauchy inequality we have

$$\frac{1}{j!} \left| \frac{\partial^{j} F(z^{*} + t^{*} \mathbf{b})}{\partial \mathbf{b}^{j}} \right| \leq \left( \frac{L(z^{*} + t^{*} \mathbf{b})}{r_{1}} \right)^{j} |F(z^{*} + t_{0} \mathbf{b})|, j \in \mathbb{Z}_{+}$$
(21)

and

$$\left| \frac{\partial^{j} F(z^{*} + t_{0j} \mathbf{b})}{\partial \mathbf{b}^{j}} - \frac{\partial^{j} F(z^{*} + t^{*} \mathbf{b})}{\partial \mathbf{b}^{j}} \right| = \left| \int_{t^{*}}^{t_{0j}} \frac{\partial^{j+1} F(z^{*} + t \mathbf{b})}{\partial \mathbf{b}^{j+1}} dt \right| \leq \left| \frac{\partial^{j+1} F(z^{*} + t_{0(j+1)} \mathbf{b})}{\partial \mathbf{b}^{j+1}} \right| \frac{r_{2}}{L(z^{*} + t^{*} \mathbf{b})}.$$
(22)

The inequalities (21) and (22) imply that

$$\left| \frac{\partial^{j+1} F(z^* + t_{0(j+1)} \mathbf{b})}{\partial \mathbf{b}^{j+1}} \right| \ge \frac{L(z^* + t^* \mathbf{b})}{r_2} \left\{ \left| \frac{\partial^j F(z^* + t_{0j} \mathbf{b})}{\partial \mathbf{b}^j} \right| - \left| \frac{\partial^j F(z^* + t^* \mathbf{b})}{\partial \mathbf{b}^j} \right| \right\} \ge$$

$$\ge \frac{L(z^* + t^* \mathbf{b})}{r_2} \left| \frac{\partial^j F(z^* + t_{0j} \mathbf{b})}{\partial \mathbf{b}^j} \right| - \frac{j! L^{j+1}(z^* + t^* \mathbf{b})}{r_2(r_1)^j} |F(z^* + t_0 \mathbf{b})|, j \in \mathbb{Z}_+.$$

Hence, for  $k_0 \geq 1$  we get

$$\left| \frac{\partial^{k_0} F(z^* + t_{0k_0} \mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| \ge \frac{L(z^* + t^* \mathbf{b})}{r_2} \left| \frac{\partial^{k_0 - 1} F(z^* + t_{0(k_0 - 1)} \mathbf{b})}{\partial \mathbf{b}^{k_0 - 1}} \right| -$$

$$-\frac{(k_{0}-1)!L^{k_{0}}(z^{*}+t^{*}\mathbf{b})}{r_{2}(r_{1})^{k_{0}-1}}|F(z^{*}+t_{0}\mathbf{b})| \geq \ldots \geq \frac{L^{k_{0}}(z^{*}+t^{*}\mathbf{b})}{(r_{2})^{k_{0}}}|F(z^{*}+t_{00}\mathbf{b})| -$$

$$-\left(\frac{0!}{(r_{2})^{k_{0}}} + \frac{1!}{(r_{2})^{k_{0}-1}r_{1}} + \ldots + \frac{(k_{0}-1)!}{r_{2}(r_{1})^{k_{0}-1}}\right)L^{k_{0}}(z^{*}+t^{*}\mathbf{b}) \times$$

$$\times |F(z^{*}+t_{0}\mathbf{b})| = \frac{L^{k_{0}}(z^{*}+t^{*}\mathbf{b})}{(r_{2})^{k_{0}}}|F(z^{*}+t_{0}\mathbf{b})| \left(\frac{|F(z^{*}+t_{00}\mathbf{b})|}{|F(z^{*}+t_{0}\mathbf{b})|} - \sum_{j=0}^{k_{0}-1}j!\left(\frac{r_{2}}{r_{1}}\right)^{j}\right). \tag{23}$$

Since (5) we have  $|F(z^* + t_{00}\mathbf{b})|/|F(z^* + t_0\mathbf{b})| > P_*$ . Besides, this inequality holds

$$\sum_{j=0}^{k_0-1} j! \left(\frac{r_2}{r_1}\right)^j \le k_0! \left(\frac{(r_2/r_1)^{k_0} - 1}{r_2/r_1 - 1}\right) \le n_0! \frac{r_1}{r_2 - r_1} \left(\frac{r_2}{r_1}\right)^{n_0}.$$

Applying (20), we obtain

$$\frac{|F(z^* + t_{00}\mathbf{b})|}{|F(z^* + t_{0}\mathbf{b})|} - \sum_{i=0}^{k_0 - 1} j! \left(\frac{r_2}{r_1}\right)^j > P_* - n_0! \frac{r_1}{r_2 - r_1} \left(\frac{r_2}{r_1}\right)^{n_0} = n_0! \left(\frac{r_2}{r_1}\right)^{n_0} P_0 + 1.$$

From (23), in view of (19) and (21), it follows that

$$\left| \frac{\partial^{k_0} F(z^* + t_{0k_0} \mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| > \frac{L^{k_0} (z^* + t^* \mathbf{b})}{(r_2)^{k_0}} \left( P_* - n_0! \frac{r_1}{r_2 - r_1} \left( \frac{r_2}{r_1} \right)^{n_0} \right) \left( \frac{r_1}{L(z^* + t^* \mathbf{b})} \right)^{k_0} \times \frac{1}{k_0!} \left| \frac{\partial^{k_0} F(z^* + t^* \mathbf{b})}{\partial \mathbf{b}^{k_0}} \right| \ge \left( \frac{r_1}{r_2} \right)^{n_0} \frac{1}{n_0! P_0} \left( P_* - n_0! \frac{r_1}{r_2 - r_1} \left( \frac{r_2}{r_1} \right)^{n_0} \right) \left| \frac{\partial^{k_0} F(z^* + t_{0k_0} \mathbf{b})}{\partial \mathbf{b}^{k_0}} \right|.$$

Hence,  $P_* < n_0! \left(\frac{r_2}{r_1}\right)^{n_0} \left(P_0 + \frac{r_1}{r_2 - r_1}\right)$  and it contradicts (20).

**Sufficiency.** We choose any two numbers  $r_1 \in (0,1)$  and  $r_2 \in (1,\beta)$ . For given  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$  we expand a function  $F(z^0 + t\mathbf{b})$  in the power series by powers  $t - t_0$ 

$$F(z^{0} + t\mathbf{b}) = \sum_{m=0}^{\infty} b_{m}(z^{0} + t_{0}\mathbf{b})(t - t_{0})^{m}, \ b_{m}(z^{0} + t_{0}\mathbf{b}) = \frac{1}{m!} \frac{\partial^{m} F(z^{0} + t_{0}\mathbf{b})}{\partial \mathbf{b}^{m}}$$

in a disc  $\left\{t: |t-t_0| \leq \frac{\beta}{L(z^0+t_0\mathbf{b})}\right\} \subset S_{z^0}$ . For  $r \leq \frac{\beta}{L(z^0+t_0\mathbf{b})}$  we denote

$$M_{\mathbf{b}}(r, z^0, t_0, F) = \max\{|F(z^0 + t\mathbf{b})| : |t - t_0| = r\},\$$

$$\mu_{\mathbf{b}}(r, z^0, t_0, F) = \max\{|b_m(z^0 + t_0\mathbf{b})|r^m : m \ge 0\},\$$

$$\nu_{\mathbf{b}}(r, z^0, t_0, F) = \max\{|b_m(z^0)|r^m : |b_m(z^0 + t_0\mathbf{b})|r^m = \mu_{\mathbf{b}}(r, z^0, t_0, F)\}.$$

By Cauchy inequality  $\mu_{\mathbf{b}}(r, z^0, t_0, F) \leq M_{\mathbf{b}}(r, z^0, t_0, F)$ . But for  $r = \frac{1}{L(z^0 + t_0 \mathbf{b})}$  we have

$$M_{\mathbf{b}}(r_1 r, z^0, t_0, F) \le \sum_{m=0}^{\infty} |b_m(z^0 + t_0 \mathbf{b})| r^m r_1^m \le \mu_{\mathbf{b}}(r, z^0, t_0, F) \sum_{m=0}^{\infty} r_1^m = \frac{1}{1 - r_1} \mu_{\mathbf{b}}(r, z^0, t_0, F)$$

and, applying a monotone of  $\nu_{\mathbf{b}}(r, z^0, t_0, F)$  by r, we get

$$\ln \mu_{\mathbf{b}}(r_2r, z^0, t_0, F) - \ln \mu_{\mathbf{b}}(r, z^0, t_0, F) = \int_r^{r_2r} \frac{\nu_{\mathbf{b}}(t, z^0, t_0, F)}{t} dt \ge \nu_{\mathbf{b}}(r, z^0, t_0, F) \ln r_2.$$

Hence, we get

$$\nu_{\mathbf{b}}(r, z^{0}, t_{0}, F) \leq \frac{1}{\ln r_{2}} (\ln \mu_{\mathbf{b}}(r_{2}r, z^{0}, t_{0}, F) - \ln \mu_{\mathbf{b}}(r, z^{0}, t_{0}, F)) \leq$$

$$\leq \frac{1}{\ln r_{2}} \{\ln M_{\mathbf{b}}(r_{2}r, z^{0}, t_{0}, F) - \ln((1 - r_{1})M_{\mathbf{b}}(r_{1}r, z^{0}, t_{0}, F))\} =$$

$$= -\frac{\ln(1 - r_{1})}{\ln r_{2}} + \frac{1}{\ln r_{2}} \{\ln M_{\mathbf{b}}(r_{2}r, z^{0}, t_{0}, F) - \ln M_{\mathbf{b}}(r_{1}r, z^{0}, t_{0}, F))\}$$
(24)

Let  $N_{\mathbf{b}}(z^0 + t_0 \mathbf{b}, L, F)$  be L-index in direction of function F at a point  $z^0 + t_0 \mathbf{b}$ , i.e.  $N_{\mathbf{b}}(z^0 + t_0 \mathbf{b}, L, F)$  is the smallest number  $m_0$  for which an inequality (3) holds with  $z = z^0 + t_0 \mathbf{b}$ . It is obvious that

$$N_{\mathbf{b}}(z^0 + t_0 \mathbf{b}, L, F) \le \nu_{\mathbf{b}}(1/L(z^0 + t^0 \mathbf{b}), z^0, t_0, F) = \nu_{\mathbf{b}}(r, z^0, t_0, F).$$

However, an inequality (18) can be written in the following form

$$M_{\mathbf{b}}\left(\frac{r_2}{L(z^0 + t_0\mathbf{b})}, z^0, t_0, F\right) \le P_1(r_1, r_2) M_{\mathbf{b}}\left(\frac{r_1}{L(z^0 + t_0\mathbf{b})}, z^0, t_0, F\right).$$

Thus, from (24) we have  $N_{\mathbf{b}}(z^0 + t_0 \mathbf{b}, L, F) \leq -\frac{\ln(1-r_1)}{\ln r_2} + \frac{\ln P_1(r_1, r_2)}{\ln r_2}$  for every  $z^0 \in \mathbb{C}^n$ ,  $t_0 \in \mathbb{C}$ , i.e.

$$N_{\mathbf{b}}(F, L) \le -\frac{\ln(1 - r_1)}{\ln r_2} + \frac{\ln P_1(r_1, r_2)}{\ln r_2}.$$

Theorem 8 is proved.

In view of proof of Theorem 8 the following theorem is true.

**Theorem 9.** Let  $\beta > 1$  and  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exist numbers  $r_1$  and  $r_2$ ,  $0 < r_1 < 1 < r_2 \le \beta$ , and  $P_1 \ge 1$  that for every  $z^0 \in \mathbb{B}_n$  and  $t_0 \in S_{z^0}$  inequality (18) holds.

Here is another criterion that is an analogue of Hayman Theorem [13].

**Theorem 10.** Let  $\beta > 1$  and  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ . An analytic in  $\mathbb{B}_n$  function F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if there exist  $p \in \mathbb{Z}_+$  and C > 0 such that for every  $z \in \mathbb{B}_n$  the following inequality holds

$$\left| \frac{1}{L^{p+1}(z)} \frac{\partial^{p+1} F(z)}{\partial \mathbf{b}^{p+1}} \right| \le C \max \left\{ \left| \frac{1}{L^k(z)} \frac{\partial^k F(z)}{\partial \mathbf{b}^k} \right| : \ 0 \le k \le p \right\}.$$
 (25)

*Proof.* Necessity. If  $N_{\mathbf{b}}(F, L) < +\infty$  then by definition of L-index boundedness in the direction we obtain an inequality (25) with  $p = N_{\mathbf{b}}(F, L)$  and  $C = (N_{\mathbf{b}}(F, L) + 1)!$ 

**Sufficiency.** Let an inequality (25) holds,  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$  and

$$K = \left\{ t \in \mathbb{C} : |t - t_0| \le \frac{1}{L(z^0 + t_0 \mathbf{b})} \right\}.$$

Thus, using  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ , for every  $t \in K$  with (25) we have

$$\frac{1}{L^{p+1}(z^{0}+t_{0}\mathbf{b})} \left| \frac{\partial^{p+1}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{p+1}} \right| \leq \left( \frac{L(z^{0}+t\mathbf{b})}{L(z^{0}+t_{0}\mathbf{b})} \right)^{p+1} \frac{1}{L^{p+1}(z^{0}+t\mathbf{b})} \times \\
\times \left| \frac{\partial^{p+1}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{p+1}} \right| \leq (\lambda_{2}^{\mathbf{b}}(1))^{p+1} \frac{1}{L^{p+1}(z^{0}+t\mathbf{b})} \left| \frac{\partial^{p+1}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{p+1}} \right| \leq \\
\leq C(\lambda_{2}^{\mathbf{b}}(1))^{p+1} \max \left\{ \left| \frac{1}{L^{k}(z^{0}+t\mathbf{b})} \frac{\partial^{k}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : 0 \leq k \leq p \right\} \leq \\
\leq C(\lambda_{2}^{\mathbf{b}}(1))^{p+1} \max \left\{ \left( \frac{L(z^{0}+t_{0}\mathbf{b})}{L(z^{0}+t\mathbf{b})} \right)^{k} \left| \frac{1}{L^{k}(z^{0}+t_{0}\mathbf{b})} \frac{\partial^{k}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| : \\
0 \leq k \leq p \right\} \leq C(\lambda_{2}^{\mathbf{b}}(1))^{p+1} \max \left\{ \left| \frac{1}{L^{k}(z^{0}+t_{0}\mathbf{b})} \frac{\partial^{k}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{k}} \right| \times \\
\times (\lambda_{1}^{\mathbf{b}}(1))^{-k} : 0 \leq k \leq p \right\} \leq Bg_{z^{0}}(t_{0}, t), \tag{26}$$

where  $B = C(\lambda_2^{\mathbf{b}}(1))^{p+1}(\lambda_1^{\mathbf{b}}(1))^{-p}$  and

$$g_{z^0}(t_0,t) = \max \left\{ \left| \frac{1}{L^k(z^0 + t_0 \mathbf{b})} \frac{\partial^k F(z^0 + t \mathbf{b})}{\partial \mathbf{b}^k} \right| : 0 \le k \le p \right\}.$$

We introduce denotations

$$\gamma_1 = \left\{ t \in \mathbb{C} : |t - t_0| = \frac{1}{2\beta L(z^0 + t_0 \mathbf{b})} \right\}, \ \gamma_2 = \left\{ t \in \mathbb{C} : |t - t_0| = \frac{\beta}{L(z^0 + t_0 \mathbf{b})} \right\}.$$

We choose arbitrary points  $t_1 \in \gamma_1$ ,  $t_2 \in \gamma_2$  and join them by a piecewise-analytic curve  $\gamma = (t = t(s), 0 \le s \le T)$ , that  $g_{z^0}(t_0, t) \ne 0$  with  $t \in \gamma$ . We choose a curve  $\gamma$  that its length  $|\gamma|$  does not exceed  $\frac{2\beta^2 + 1}{\beta L(z^0 + t_0 \mathbf{b})}$ .

The function  $g_{z^0}(t_0, t(s))$  is continuous on [0, T]. Without loss of generality, we consider that function t = t(s) is analytic on [0, T]. Otherwise, we can consider separately the intervals of analyticity for this function and repeat similar arguments which below we present for [0, T]. First, we prove that the function  $g_{z^0}(t_0, t(s))$  is continuously differentiable on [0, T] except, perhaps, a finite set of points. For arbitrary  $k_1, k_2, 0 \le k_1 \le k_2 \le p$ , either

$$\frac{1}{L^{k_1}(z^0 + t_0 \mathbf{b})} \left| \frac{\partial^{k_1} F(z^0 + t(s) \mathbf{b})}{\partial \mathbf{b}^{k_1}} \right| \equiv \frac{1}{L^{k_2}(z^0 + t_0 \mathbf{b})} \left| \frac{\partial^{k_2} F(z^0 + t(s) \mathbf{b})}{\partial \mathbf{b}^{k_2}} \right|$$

or the equality

$$\frac{1}{L^{k_1}(z^0 + t_0 \mathbf{b})} \left| \frac{\partial^{k_1} F(z^0 + t(s) \mathbf{b})}{\partial \mathbf{b}^{k_1}} \right| = \frac{1}{L^{k_2}(z^0 + t_0 \mathbf{b})} \left| \frac{\partial^{k_2} F(z^0 + t(s) \mathbf{b})}{\partial \mathbf{b}^{k_2}} \right|$$

holds only for a finite set of points  $s_k \in [0, T]$ . Thus, we can split the segment [0, T] on a finite number of segments that on each segment

$$g_{z^0}(t_0, t(s)) \equiv \frac{1}{L^k(z^0 + t_0 \mathbf{b})} \left| \frac{\partial^k F(z^0 + t(z) \mathbf{b})}{\partial \mathbf{b}^k} \right|$$

for some  $k, 0 \le k \le p$ . This means that a function  $g_{z^0}(t_0, t(s))$  is continuously differentiable except, perhaps, a finite set of points. Since (26) we obtain

$$\frac{dg_{z^{0}}(t_{0}, t(s))}{ds} \leq \max \left\{ \frac{d}{ds} \left( \frac{1}{L^{k}(z^{0} + t_{0}\mathbf{b})} \left| \frac{\partial^{k} F(z^{0} + t(s)\mathbf{b})}{\partial \mathbf{b}^{k}} \right| \right) : 0 \leq k \leq p \right\} \leq \\
\leq \max \left\{ \frac{1}{L^{k}(z^{0} + t_{0}\mathbf{b})} \left| \frac{\partial^{k+1} F(z^{0} + t(s)\mathbf{b})}{\partial \mathbf{b}^{k+1}} \right| |t'(s)| : 0 \leq k \leq p \right\} = \\
= L(z^{0} + t_{0}\mathbf{b})|t'(s)| \max \left\{ \frac{1}{L^{k+1}(z^{0} + t_{0}\mathbf{b})} \left| \frac{\partial^{k+1} F(z^{0} + t(s)\mathbf{b})}{\partial \mathbf{b}^{k+1}} \right| : 0 \leq k \leq p \right\} \leq \\
\leq Bg_{z^{0}}(t_{0}, t(s))|t'(s)|L(z^{0} + t_{0}\mathbf{b}).$$

Hence, we have

$$\left| \ln \frac{g_{z^0}(t_0, t_2)}{g_{z^0}(t_0, t_1)} \right| = \left| \int_0^T \frac{dg_{z^0}(t_0, t(s))}{g_{z^0}(t_0, t(s))} \right| \le BL(z^0 + t_0 \mathbf{b}) \int_0^T |t'(s)| ds =$$

$$= BL(z^0 + t_0 \mathbf{b}) |\gamma| \le 2B \frac{\beta^2 + 1}{\beta}.$$

If we pick a point  $t_2 \in \gamma_2$ , for which

$$|F(z^0 + t_2 \mathbf{b})| = \max \left\{ |F(z^0 + t\mathbf{b})| : |t - t_0| = \frac{\beta}{L(z^0 + t_0 \mathbf{b})} \right\},$$

then we have

$$\max\left\{|F(z^0+t\mathbf{b})|:|t-t_0|=\frac{2}{L(z^0+t_0\mathbf{b})}\right\} \le g_{z^0}(t_0,t_2) \le g_{z^0}(t_0,t_1)\exp\{2B\frac{\beta^2+1}{\beta}\}.$$
 (27)

Applying Cauchy inequality and using  $t_1 \in \gamma_1$ , for all  $j = 1, \ldots, p$  we have

$$\left| \frac{\partial^{j} F(z^{0} + t_{1} \mathbf{b})}{\partial \mathbf{b}^{j}} \right| \leq j! (2\beta L(z^{0} + t_{0} \mathbf{b}))^{j} \max \left\{ |F(z^{0} + t\mathbf{b}) : |t - t_{1}| = \frac{1}{2\beta L(z^{0} + t_{0} \mathbf{b})} \right\} \leq j! (2\beta L(z^{0} + t_{0} \mathbf{b}))^{j} \max \left\{ |F(z^{0} + t\mathbf{b}) : |t - t_{0}| = \frac{1}{\beta L(z^{0} + t_{0} \mathbf{b})} \right\},$$

i.e.

$$g_{z^0}(t_0, t_1) \le p!(2\beta)^p \max \left\{ |F(z^0 + t\mathbf{b}) : |t - t_0| = \frac{1}{\beta L(z^0 + t_0\mathbf{b})} \right\}.$$

Thus, (27) implies

$$|F(z^0 + t_2 \mathbf{b})| = \max \left\{ |F(z^0 + t\mathbf{b})| : |t - t_0| = \frac{\beta}{L(z^0 + t_0 \mathbf{b})} \right\} \le g_{z^0}(t_0, t_2) \le$$

$$\leq g_{z^0}(t_0, t_1) \exp\left\{2B \frac{\beta^2 + 1}{\beta}\right\} \leq p! (2\beta)^p \exp\left\{2B \frac{\beta^2 + 1}{\beta}\right\} \times \max\left\{|F(z^0 + t\mathbf{b})| : |t - t_0| = \frac{1}{\beta L(z^0 + t_0\mathbf{b})}\right\}.$$

By Theorem 9 this inequality implies that a function F is of bounded L-index in the direction  $\mathbf{b} \in \mathbb{C}^n$ . Theorem 10 is proved.

The following theorem gives an estimate of maximum modulus by minimum modulus.

**Theorem 11.** Let  $\beta > 1$  and  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ . Analytic in  $\mathbb{B}_n$  function F(z) is of bounded Lindex in direction  $\mathbf{b}$  if and only if for every R,  $0 < R \le \beta$ , there exist numbers  $P_2(R) \ge 1$  and  $\eta(R) \in (0,R)$  that for each  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$  and some  $r = r(z^0,t_0) \in [\eta(R),R]$  the following
inequality is valid

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{2} \min \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r}{L(z^{0} + t_{0}\mathbf{b})} \right\}.$$
(28)

*Proof.* Necessity. Let  $N_{\mathbf{b}}(F, L) = N < +\infty$  and  $R \geq 0$ . We put

$$R_0 = 1, r_0 = \frac{R}{8(R+1)}, \ R_j = \frac{R_{j-1}}{4N} r_{j-1}^N, \ r_j = \frac{1}{8} R_j (j=1,2,\dots,N).$$

Let  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$  and  $N_0 = N_{\mathbf{b}}(z^0 + t_0\mathbf{b}, L, F)$  be L-index in direction **b** of function F at point  $z^0 + t_0\mathbf{b}$ , i.e.  $N_{\mathbf{b}}(z^0 + t_0\mathbf{b}, L, F)$  is the smallest number  $m_0$ , for which inequality (3) holds with  $z = z^0 + t_0\mathbf{b}$ . The maximum in the right part of (3) is attained at  $m_0$ . But  $0 \le N_0 \le N$ . For given  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$  a function  $F(z^0 + t\mathbf{b})$  expands in power series by powers  $t - t_0$ 

$$F(z^{0} + t\mathbf{b}) = \sum_{m=0}^{\infty} b_{m}(z^{0} + t_{0}\mathbf{b})(t - t_{0})^{m}, \ b_{m}(z^{0} + t_{0}\mathbf{b}) = \frac{1}{m!} \frac{\partial^{m} F(z^{0} + t_{0}\mathbf{b})}{\partial \mathbf{b}^{m}}.$$

We put

$$a_m(z^0) = \frac{|b_m(z^0 + t_0 \mathbf{b})|}{L^m(z^0)} = \frac{1}{m!L^m(z^0)} \left| \frac{\partial^m F(z^0 + t_0 \mathbf{b})}{\partial \mathbf{b}^m} \right|.$$

For any  $m \in \mathbb{Z}_+$  inequality holds

$$a_{N_0}(z^0) \ge a_m(z^0) = R_0 a_m(z^0).$$

There exists the smallest number  $n_0 \in \{0, 1, ..., N_0\}$  that for all  $m \in \mathbb{Z}_+$   $a_{n_0}(z^0) \ge a_m(z^0) R_{N_0 - n_0}$ . Thus,  $a_{n_0}(z^0) \ge a_{N_0}(z^0) R_{N_0 - n_0}$  and  $a_j(z^0) < a_{N_0}(z^0) R_{N_0 - j}$  for  $j < n_0$ , because if  $a_{j_0}(z^0) \ge a_{N_0}(z^0) R_{N_0 - j_0}$  for some  $j_0 < n_0$ , then  $a_{j_0}(z^0) \ge a_m(z^0) R_{N_0 - j_0}$  for all  $m \in \mathbb{Z}_+$  and it contradicts the choice of  $n_0$ . Since inequalities  $a_j(z^0) < a_{N_0}(z^0) R_{N_0 - j}$  ( $j < n_0$ ) and  $a_m(z^0) \le a_{N_0}(z^0)$  ( $m > n_0$ ) for  $t \in S_{z^0}$  and  $|t - t_0| = \frac{1}{L(z^0 + t_0 \mathbf{b})} r_{N_0 - n_0}$  we have

$$|F(z^0 + t\mathbf{b})| = |b_{n_0}(z^0 + t_0\mathbf{b})(t - t_0)^{n_0} + \sum_{m \neq n_0} b_m(z^0 + t_0\mathbf{b})(t - t_0)^m| \ge$$

$$\geq |b_{n_0}(z^0)||t - t_0|^{n_0} - \sum_{m \neq n_0} |b_m(z^0)||t - t_0|^m = a_{n_0}(z^0) r_{N_0 - n_0}^{n_0} - \sum_{m \neq n_0} a_m(z^0) r_{N_0 - n_0}^m = a_{n_0}(z^0) r_{N_0 - n_0}^{n_0} - \sum_{j < n_0} a_j(z^0) r_{N_0 - n_0}^j - \sum_{j < n_0} a_j(z^0) r_{N_0 - n_0}^j - \sum_{j < n_0} a_{N_0}(z^0) r_{N_0 - n_0}^m \geq a_{N_0}(z^0) R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - \sum_{j < n_0} a_{N_0}(z^0) r_{N_0 - n_0}^m \geq a_{N_0}(z^0) R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - n_0 a_{N_0}(z^0) R_{N_0 - n_0 + 1} - \sum_{m > n_0} a_{N_0}(z^0) r_{N_0 - n_0}^{n_0 + 1} \frac{1}{1 - r_{N_0 - n_0}} = a_{N_0}(z^0) \left( R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - \frac{n_0}{4N} R_{N_0 - n_0} r_{N_0 - n_0}^N - r_{N_0 - n_0}^{n_0} - \frac{1}{4} R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - \frac{1}{4} R_$$

Besides, for  $t \in S_{z^0}$  the following inequality holds

$$|F(z^{0} + t\mathbf{b})| \leq \sum_{m=0}^{+\infty} |b_{m}(z^{0} + t_{0}\mathbf{b})||t - t_{0}|^{m} = \sum_{m=0}^{\infty} a_{m}(z^{0})r_{N_{0} - n_{0}}^{m} \leq$$

$$\leq a_{N_{0}}(z^{0}) \sum_{m=0}^{+\infty} r_{N_{0} - n_{0}}^{m} = \frac{a_{N_{0}}(z^{0})}{1 - r_{N_{0} - n_{0}}} \leq \frac{a_{N_{0}}(z^{0})}{1 - 1/8} = \frac{8}{7}a_{N_{0}}(z^{0}).$$
(30)

From (29) and (30) we have

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r_{N_{0} - n_{0}}}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le \frac{8}{7} a_{N_{0}}(z^{0}) \le$$

$$\le \frac{16}{7} \frac{1}{R_{N_{0} - n_{0}}} r_{N_{0} - n_{0}}^{-n_{0}} \min \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r_{N_{0} - n_{0}}}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le \frac{16}{7} \frac{1}{R_{N}} r_{N}^{-N} \min \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r_{N_{0} - n_{0}}}{L(z^{0} + t_{0}\mathbf{b})} \right\},$$

i.e. (28) holds with  $P_2(R) = \frac{16}{7R_N r_N^N}$ ,  $\eta(R) = r_N = \frac{1}{8R_N}$  and  $r = r_{N_0 - n_0}$ .

**Sufficiency.** In view of Theorem 9 it is enough to prove there exists number  $P_1$  that for every  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$ 

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\beta + 1}{2L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{1} \max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\beta - 1}{4\beta L(z^{0} + t_{0}\mathbf{b})} \right\}. \quad (31)$$

Let  $\widetilde{R} = \frac{\beta - 1}{4\beta}$ . Then there exist  $P_2^* = P_2(\widetilde{R})$  and  $\eta = \eta(\widetilde{R}) \in (0, \widetilde{R})$  that for every  $z^* \in \mathbb{B}_n$ ,  $t^* \in S_{z^*}$  and some  $r \in [\eta, \widetilde{R}]$  the following inequality is valid

$$\max \{ |F(z^* + t\mathbf{b})| : |t - t^*| = \frac{r}{L(z^0 + t^*\mathbf{b})} \} \le$$

$$\leq P_2^* \min \{ |F(z^* + t\mathbf{b})| : |t - t^*| = \frac{r}{L(z^0 + t^*\mathbf{b})} \}.$$

Let  $L^* = \max\{L(z^0 + t\mathbf{b}) : |t - t_0| \le \beta/L(z^0 + t_0\mathbf{b})\}$ ,  $\rho_0 = (\beta - 1)/(4\beta L(z^0 + t_0\mathbf{b}))$ ,  $\rho_k = \rho_0 + k\eta/L^*$ ,  $k \in \mathbb{Z}_+$ . Hence,  $\frac{\eta}{L^*} < \frac{\beta - 1}{4\beta L(z^0 + t_0\mathbf{b})} < \frac{\beta}{L(z^0 + t_0\mathbf{b})} - \frac{\beta + 1}{2L(z^0 + t_0\mathbf{b})}$ . Therefore, there exists  $n^* \in \mathbb{N}$ , which does not depend on  $z^0$  and  $t_0$  that  $\rho_{p-1} < \frac{\beta + 1}{2L(z^0 + t_0\mathbf{b})} \le \rho_p \le \frac{\beta}{L(z^0 + t_0\mathbf{b})}$  for some  $p = p(z^0, t_0) \le n^*$ .

Let  $c_k = \{t \in \mathbb{C} : |t - t_0| = \rho_k\}, |F(z^0 + t_k^{**}\mathbf{b})| = \max\{|F(z^0 + t\mathbf{b})| : t \in c_k\}$  and  $t_k^*$  be a point of intersection of the segment  $[t_0, t_k^{**}]$  with the circle  $c_{k-1}$ . Then for every  $r > \eta$  the following inequality holds  $|t_k^{**} - t_k^{*}| = \eta/L^* \le r/L(z^0 + t_k^{*}\mathbf{b})$ . Hence, for some  $r \in [\eta, \widetilde{R}]$  the following inequality is valid

$$\begin{split} |F(z^0 + t_k^{**}\mathbf{b})| &\leq \max \left\{ |F(z^0 + t\mathbf{b})| : |t - t_k^{*}| = \frac{r}{L(z^0 + t_k^{*}\mathbf{b})} \right\} \leq \\ &\leq P_2^* \min \left\{ |F(z^0 + t\mathbf{b})| : |t - t_k^{*}| = \frac{r}{L(z^0 + t_k^{*}\mathbf{b})} \right\} \leq P_2^* \max\{ |F(z^0 + t\mathbf{b})| : t \in c_{k-1} \}. \end{split}$$

Therefore, we get inequality (31) with  $P_1^* = (P_2^*)^{n^*}$ 

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\beta + 1}{2L(z^{0} + t_{0}\mathbf{b})} \right\} \leq \max\{|F(z^{0} + t\mathbf{b})| : t \in c_{p}\} \leq \\
\leq P_{2}^{*} \max\{|F(z^{0} + t\mathbf{b})| : t \in c_{p-1}\} \leq \dots \leq (P_{2}^{*})^{p} \max\{|F(z^{0} + t\mathbf{b})| : t \in c_{0}\} \leq \\
\leq (P_{2}^{*})^{n^{*}} \max\left\{|F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\beta - 1}{4\beta L(z^{0} + t_{0}\mathbf{b})} \right\}.$$

Theorem 11 is proved.

#### 6 Logarithmic derivative and zeros.

Below we prove another criterion of L-index boundedness in a direction that describes behaviour of the directional logarithmic derivative and distribution of zeros.

We need some additional denotations.

Denote  $g_{z^0}(t) := F(z^0 + t\mathbf{b})$ . If for a given  $z^0 \in \mathbb{B}_n$   $g_{z^0}(t) \neq 0$  for all  $t \in S_{z^0}$ , then  $G_r^{\mathbf{b}}(F, z^0) := \emptyset$ ; if for a given  $z^0 \in \mathbb{B}_n$   $g_{z^0}(t) \equiv 0$ , then  $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b} : t \in S_{z^0}\}$ . And if for a given  $z^0 \in \mathbb{B}_n$   $g_{z^0}(t) \not\equiv 0$  and  $a_k^0$  are zeros of  $g_{z^0}(t)$ , then

$$G_r^{\mathbf{b}}(F, z^0) := \bigcup_k \left\{ z^0 + t\mathbf{b} \colon |t - a_k^0| \le \frac{r}{L(z^0 + a_k^0 \mathbf{b})} \right\}, \quad r > 0.$$

Let

$$G_r^{\mathbf{b}}(F) = \bigcup_{z^0 \in \mathbb{B}_n} G_r^{\mathbf{b}}(F, z^0). \tag{32}$$

We remark that if  $L(z) \equiv 1$ , then  $G_r^{\mathbf{b}}(F) \subset \{z \in \mathbb{B}_n : dist(z, \mathbb{Z}_F) < r|\mathbf{b}|\}$ , where  $\mathbb{Z}_F$  is a zero set of function F. By  $n(r, z^0, t_0, 1/F) = \sum_{|a_k^0 - t_0| \le r} 1$  we denote a counting function of zeros  $a_k^0$ .

**Theorem 12.** Let F(z) be an analytic in  $\mathbb{B}_n$  function,  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$  and  $\mathbb{B}_n \setminus G_{\beta}^{\mathbf{b}}(F) \neq \emptyset$ . F(z) is of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$  if and only if

1) for every  $r \in (0, \beta]$  there exists P = P(r) > 0 that for each  $z \in \mathbb{B}_n \backslash G_r^{\mathbf{b}}(F)$ 

$$\left| \frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{b}} \right| \le PL(z); \tag{33}$$

2) for every  $r \in (0, \beta]$  there exists  $\widetilde{n}(r) \in \mathbb{Z}_+$  that for each  $z^0 \in \mathbb{B}_n$  with  $F(z^0 + t\mathbf{b}) \not\equiv 0$ , and for each  $t_0 \in S_{z^0}$ 

$$n\left(\frac{r}{L(z^0 + t^0\mathbf{b})}, z^0, t_0, \frac{1}{F}\right) \le \widetilde{n}(r). \tag{34}$$

*Proof.* Necessity. First, we prove that if function F(z) is of bounded L-index in a direction, then for every  $\widetilde{z}^0 = z^0 + t_0 \mathbf{b} \in \mathbb{B}_n \backslash G_r^{\mathbf{b}}(F)$   $(r \in (0, \beta])$  and for every  $\widetilde{a}^k = z^0 + a_k^0 \mathbf{b}$  the following inequality holds

$$|\widetilde{z}^0 - \widetilde{a}_k| > \frac{r|\mathbf{b}|}{2L(\widetilde{z}^0)\lambda_2^{\mathbf{b}}(z^0, r)}.$$
(35)

On the contrary, we assume that there exists  $\widetilde{z}^0 = z^0 + t_0 \mathbf{b} \in \mathbb{B}_n \backslash G_r^{\mathbf{b}}(F)$  and  $\widetilde{a}^k = z^0 + a_k^0 \mathbf{b}$  that

$$|\widetilde{z}^0 - \widetilde{a}_k| \le \frac{r|\mathbf{b}|}{2L(\widetilde{z}^0)\lambda_2^{\mathbf{b}}(z^0, r)} \le \frac{r|\mathbf{b}|}{2L(\widetilde{z}^0)} < \frac{r|\mathbf{b}|}{L(\widetilde{z}^0)}.$$

Hence,  $|t_0 - a_k^0| < \frac{r}{L(\tilde{z}^0)}$ . But for  $\lambda_2^{\mathbf{b}}$  the following estimate holds

$$L(\widetilde{a}^k) \le \lambda_2^{\mathbf{b}}(z^0, r) L(\widetilde{z}^0),$$

and therefore

$$|\widetilde{z}^0 - \widetilde{a}^k| = |\mathbf{b}| \cdot |t_0 - a_k^0| \le \frac{r|\mathbf{b}|}{2L(\widetilde{a}^k)},$$

i.e.  $|t_0 - a_k^0| \leq \frac{r}{2L(\widetilde{a}^k)}$ . We obtained a contradiction with  $\widetilde{z}^0 \in \mathbb{C}^n \backslash G_r^{\mathbf{b}}(F)$ . In fact, in (35) instead of  $\lambda_2^{\mathbf{b}}(z^0, r)$  we can take  $\lambda_2^{\mathbf{b}}(r)$ .

We choose in Theorem 11  $R = \frac{r}{2\lambda_2^{\mathbf{b}}(r)}$ . Then there exists  $P_2 \geq 1$  and  $\eta \in (0, R)$  that for every  $\widetilde{z^0} = z^0 + t_0 \mathbf{b} \in \mathbb{B}_n$  and some  $r^* \in [\eta, R]$  inequality (28) holds with  $r^*$  instead of r. Therefore, by Cauchy inequality

$$\left| \frac{\partial F(z^{0} + t_{0}\mathbf{b})}{\partial \mathbf{b}} \right| \leq \frac{L(z^{0} + t_{0}\mathbf{b})}{r^{*}} \max \left\{ |F(z^{0} + t\mathbf{b}) : |t - t_{0}| = \frac{r^{*}}{L(z^{0} + t_{0}\mathbf{b})} \right\} \leq 
\leq P_{2} \frac{L(z^{0} + t_{0}\mathbf{b})}{\eta} \min \{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r^{*}}{L(z^{0} + t_{0}\mathbf{b})} \right\}$$
(36)

Since (35) for every  $z^0 + t_0 \mathbf{b} \in \mathbb{B}_n \backslash G_r^{\mathbf{b}}(F)$ , a set

$$\left\{ z^{0} + t\mathbf{b} : |t - t_{0}| \le \frac{r}{2\lambda_{2}^{\mathbf{b}}(r) L(z^{0} + t_{0}\mathbf{b})} \right\}$$

does not contain zeros of function  $F(z^0+t\mathbf{b})$ . Therefore, applying to 1/F, as a function of variable t, a maximum principle, we have

$$|F(z^{0} + t_{0}\mathbf{b})| \ge \min\left\{|F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r^{*}}{L(z^{0} + t_{0}\mathbf{b})}\right\}$$
 (37)

The inequalities (36) and (37) imply (33) with  $P = \frac{P_2}{n}$ .

Now we prove that if F is of bounded L-index in direction  $\mathbf{b}$  then there exists  $P_3 > 0$  that for every  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$ ,  $r \in (0,1]$ 

$$n\left(\frac{r}{L(z^{0}+t_{0}\mathbf{b})}, z^{0}, t_{0}, 1/F\right) \min\left\{|F(z^{0}+t\mathbf{b})| : |t-t_{0}| = \frac{r}{L(z^{0}+t_{0}\mathbf{b})}\right\} \leq$$

$$\leq P_{3} \max\left\{|F(z^{0}+t\mathbf{b})| : |t-t_{0}| = \frac{1}{L(z^{0}+t_{0}\mathbf{b})}\right\}.$$
(38)

By Cauchy inequality and Theorem 8 for all  $t \in S_{z^0}$  with circle  $|t - t_0| = \frac{1}{L(z^0 + t_0 \mathbf{b})}$  we have

$$\left| \frac{\partial F(z^{0} + t\mathbf{b})}{\partial \mathbf{b}} \right| \leq \frac{L(z^{0} + t_{0}\mathbf{b})}{\beta - 1} \max \left\{ |F(z^{0} + \theta\mathbf{b})| : |\theta - t| = \frac{\beta - 1}{L(z^{0} + t_{0}\mathbf{b})} \right\} \leq 
\leq \frac{L(z^{0} + t_{0}\mathbf{b})}{\beta - 1} \max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\beta}{L(z^{0} + t_{0}\mathbf{b})} \right\} \leq 
\leq \frac{P_{1}(1, \beta)}{\beta - 1} L(z^{0} + t_{0}\mathbf{b}) \max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{1}{L(z^{0} + t_{0}\mathbf{b})} \right\}.$$
(39)

If  $F(z^0 + t\mathbf{b}) \neq 0$  on a circle  $\left\{ t \in S_{z^0} : |t - t_0| = \frac{r}{L(z^0 + t_0\mathbf{b})} \right\}$ , then

$$n\left(\frac{r}{L(z^{0}+t_{0}\mathbf{b})}, z^{0}, t_{0}, \frac{1}{F}\right) = \left|\frac{1}{2\pi i} \int_{|t-t_{0}| = \frac{r}{L(z^{0}+t_{0}\mathbf{b})}} \frac{\partial F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}} \frac{1}{F(z^{0}+t\mathbf{b})} dt\right| \leq \frac{\max\left\{\left|\frac{\partial F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}}\right| : |t-t_{0}| = \frac{r}{L(z^{0}+t_{0}\mathbf{b})}\right\}}{\min\left\{|F(z^{0}+t\mathbf{b})| : |t-t_{0}| = \frac{r}{L(z^{0}+t_{0}\mathbf{b})}\right\}} \frac{r}{L(z^{0}+t_{0}\mathbf{b})}.$$
(40)

From (39) and (40) we have

$$n\left(\frac{r}{L(z^{0}+t_{0}\mathbf{b})},z^{0},t_{0},1/F\right)\min\left\{\left|F(z^{0}+t\mathbf{b})\right|:\left|t-t_{0}\right|=\frac{r}{L(z^{0}+t_{0}\mathbf{b})}\right\}\leq$$

$$\leq\frac{r}{L(z^{0}+t_{0}\mathbf{b})}\max\left\{\left|\frac{\partial F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}}\right|:\left|t-t_{0}\right|=\frac{r}{L(z^{0}+t_{0}\mathbf{b})}\right\}\leq$$

$$\leq\frac{1}{L(z^{0}+t_{0}\mathbf{b})}\max\left\{\left|\frac{\partial F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}}\right|:\left|t-t_{0}\right|=\frac{1}{L(z^{0}+t_{0}\mathbf{b})}\right\}\leq$$

$$\leq\frac{P_{1}(1,\beta)}{\beta-1}\max\left\{\left|F(z^{0}+t\mathbf{b})\right|:\left|t-t_{0}\right|=\frac{1}{L(z^{0}+t_{0}\mathbf{b})}\right\}.$$

Thus, we obtain (38) with  $P_3 = \frac{P_1(1,\beta)}{\beta-1}$ . If function  $F(z^0+t\mathbf{b})$  has zeros on the circle  $\left\{t \in D_R^{z^0}: |t-t_0| = \frac{r}{L(z^0+t_0\mathbf{b})}\right\}$  then an inequality (38) is obvious.

Now we put R=1 in Theorem 11. Then there exists  $P_2=P_2(1)\geq 1$  and  $\eta\in(0,1)$  that for each  $z^0\in\mathbb{B}_n,\,t_0\in S_{z^0}$  and some  $r^*=r^*(z^0,t_0)\in[\eta,1]$ 

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r^{*}}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{2} \min \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r^{*}}{L(z^{0} + t_{0}\mathbf{b})} \right\}.$$

Besides, by Theorem 8 there exists  $P_1 \geq 1$  such that for all  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$ 

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{1}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{1}(1, \eta) \max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\eta}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{1}(1, \eta) \max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r^{*}}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{1}(1, \eta) P_{2} \min \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r^{*}}{L(z^{0} + t_{0}\mathbf{b})} \right\}.$$

Since (38), we have

$$n\left(\frac{r^*}{L(z^0 + t_0\mathbf{b})}, z^0, t_0, \frac{1}{F}\right) \min\left\{ |F(z^0 + t\mathbf{b})| : |t - t_0| = \frac{r^*}{L(z^0 + t_0\mathbf{b})} \right\} \le$$

$$\le P_3 P_1(1, \eta) P_2 \min\left\{ |F(z^0 + t\mathbf{b})| : |t - t_0| = \frac{r^*}{L(z^0 + t_0\mathbf{b})} \right\},$$

i.e.  $n\left(\frac{r^*}{L(z^0+t_0\mathbf{b})}, z^0, t_0, \frac{1}{F}\right) \leq P_1(1,\eta)P_2P_3$ . Hence,

$$n\left(\frac{r^*}{L(z^0+t_0\mathbf{b})}, z^0, t_0, \frac{1}{F}\right) \le P_4 = P_1(1,\eta)P_2P_3 = \frac{P_1(1,\eta)P_2(1)P_1(1,r+1)}{r}.$$

If  $r \in (0, \eta]$  then property (34) is proved.

Let  $r \in (\eta, \beta]$  and  $L^* = \max \left\{ L(z^0 + t\mathbf{b}) : |t - t_0| = \frac{r}{L(z^0 + t_0 \mathbf{b})} \right\}$ . Using properties of  $Q_{\mathbf{b}}^n$ , we have  $L^* \leq \lambda_2^{\mathbf{b}}(r)L(z^0 + t_0 \mathbf{b})$ . Put  $\rho = \frac{\eta}{L(z^0 + t_0 \mathbf{b})\lambda_2^{\mathbf{b}}(r)}$ ,  $R = \frac{r}{L(z^0 + t_0 \mathbf{b})}$ . We can cover every set  $\overline{K} = \{z^0 + t\mathbf{b} : |t - t_0| \leq R\}$  by a finite number m = m(r) of closed sets  $\overline{K}_j = \{z^0 + t\mathbf{b} : |t - t_j| \leq \rho\}$ , where  $t_j \in \overline{K}$ . Since

$$\frac{\eta}{\lambda_2^{\mathbf{b}}(r)L(z^0 + t_0\mathbf{b})} \le \frac{\eta}{L^*} \le \frac{\eta}{L(z^0 + t_j\mathbf{b})}$$

in each  $\overline{K}_j$  there are at most  $[P_4]$  zeros of function  $F(z^0 + t\mathbf{b})$ . Thus,

$$n\left(\frac{r}{L(z^0 + t_0 \mathbf{b})}, z^0, t_0, 1/F\right) \le \widetilde{n}(r) = [P_4] m(r)$$

and property (34) is proved.

**Sufficiency.** On the contrary, suppose that conditions (33) and (34) hold. By condition (34) for every  $R \in (0, \beta]$  there exists  $\widetilde{n}(R) \in \mathbb{Z}_+$  that in each set

$$\overline{K} = \left\{ z^0 + t\mathbf{b} : |t - t_0| \le \frac{R}{L(z^0 + t_0\mathbf{b})} \right\}$$

the number of zeros of  $F(z^0 + t\mathbf{b})$  does not exceed  $\widetilde{n}(r)$ .

We put  $a = a(R) = \frac{R\lambda_1^{\mathbf{b}}(R)}{2(\widetilde{n}(R)+1)}$ . By condition (33) there exists  $P = P(a) = \widetilde{P}(R) \ge 1$  that  $\left|\frac{\partial F(z)}{\partial \mathbf{b}} \frac{1}{F(z)}\right| \le PL(z)$  for all  $z \in \mathbb{B}_n \backslash G_a^{\mathbf{b}}$ , that is for all  $z \in \overline{K}$  lying outside the sets

$$b_k^0 = \left\{ z^0 + t\mathbf{b} : |t - a_k^0| < \frac{a(R)}{L(z^0 + a_k^0\mathbf{b})} \right\},$$

where  $a_k^0 \in \overline{K}$  are zeros of function  $F(z^0 + t\mathbf{b}) \not\equiv 0$ . Since properties  $\lambda_1^{\mathbf{b}}$  we have

$$\lambda_1^{\mathbf{b}}(R)L(z^0 + t_0\mathbf{b}) \le \lambda_1^{\mathbf{b}}(R, z^0)L(z^0 + t_0\mathbf{b}) \le L(z^0 + a_k^0\mathbf{b}).$$

Therefore,  $\left| \frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{b}} \right| \leq PL(z)$  for all  $z \in \mathbb{B}_n$ , lying outside the sets

$$c_k^0 = \left\{ z^0 + t\mathbf{b} : |t - a_k^0| \le \frac{a(R)}{\lambda_1^{\mathbf{b}}(R)L(z^0 + t_0\mathbf{b})} = \frac{R}{2(\widetilde{n}(R) + 1)L(z^0 + t_0\mathbf{b})} \right\}.$$

Obviously, the sum of diameters of sets  $c_k^0$  does not exceed

$$\frac{R\widetilde{n}(R)}{(\widetilde{n}(R)+1)L(z^0+t_0\mathbf{b})} < \frac{R}{L(z^0+t_0\mathbf{b})}.$$

Therefore, there exist a set  $\tilde{c}^0 = \left\{ z^0 + t\mathbf{b} : |t - t_0| = \frac{r}{L(z^0 + t_0\mathbf{b})} \right\}$ , where

$$\frac{R}{2(\widetilde{n}(R)+1)} = \eta(R) < r < R,$$

such that for all  $z \in \tilde{c}^0$  the following inequality is valid

$$\left| \frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{b}} \right| \le PL(z) \le P\lambda_2^{\mathbf{b}}(r)L(z^0 + t_0\mathbf{b}) \le P\lambda_2^{\mathbf{b}}(R)L(z^0 + t_0\mathbf{b}).$$

For any points  $z_1 = z^0 + t_1 \mathbf{b}$  and  $z_2 = z^0 + t_2 \mathbf{b}$  with  $\tilde{c}^0$  we have

$$\ln \left| \frac{F(z^{0} + t_{1}\mathbf{b})}{F(z^{0} + t_{2}\mathbf{b})} \right| \leq \int_{t_{1}}^{t_{2}} \left| \frac{1}{F(z^{0} + t\mathbf{b})} \frac{\partial F(z^{0} + t\mathbf{b})}{\partial \mathbf{b}} \right| |dt| \leq 
\leq P\lambda_{2}^{\mathbf{b}}(R) L(z^{0} + t_{0}\mathbf{b}) \frac{2r}{L(z^{0} + t_{0}\mathbf{b})} \leq 2R P(R)\lambda_{2}^{\mathbf{b}}(R).$$

Hence, we get

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r}{L(z^{0} + t_{0}\mathbf{b})} \right\} \le$$

$$\le P_{2}\min \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{r}{L(z^{0} + t_{0}\mathbf{b})} \right\},$$

where  $P_2 = \exp\{2R P(R)\lambda_2^{\mathbf{b}}(R)\}$ . Thus, by Theorem 11 the function F(z) is of bounded L-index in direction  $\mathbf{b}$ . Theorem 12 is proved.

## 7 Boundedness *L*-index in the direction of analytical solutions of some partial differential equations.

We consider a partial differential equation

$$g_0(z)\frac{\partial^p w}{\partial \mathbf{b}^p} + g_1(z)\frac{\partial^{p-1} w}{\partial \mathbf{b}^{p-1}} + \ldots + g_p(z)w = h(z). \tag{41}$$

First, we prove an auxiliary assertion.

**Lemma 6.** Let  $\beta > 1$ ,  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ , F(z) be an analytic in  $\mathbb{B}_n$  function of bounded L-index in direction  $\mathbf{b} \in \mathbb{C}^n$ ,  $\mathbb{B}_n \backslash G^{\mathbf{b}}_{\beta}(F) \neq \emptyset$ . Then for every  $r \in (0,\beta]$  and for every  $m \in \mathbb{N}$  there exists P = P(r,m) > 0 such that for all  $z \in \mathbb{B}_n \backslash G^{\mathbf{b}}_r(F)$  inequality holds

$$\left| \frac{\partial^m F(z)}{\partial \mathbf{b}^m} \right| \le PL^m(z)|F(z)|.$$

*Proof.* In Theorem 12 we proved that if an entire function F(z) is of bounded L-index in direction  $\mathbf{b}$ , then (35) holds, i.e. for each  $\widetilde{z}^0 = z^0 + t_0 \mathbf{b} \in \mathbb{B}_n \backslash G_r^{\mathbf{b}}(F)$   $(r \in (0, \beta])$  and  $\widetilde{a}^k = z^0 + a_k^0 \mathbf{b}$  an inequality holds

$$|\widetilde{z}^0 - \widetilde{a}_k| > \frac{r|\mathbf{b}|}{2L(\widetilde{z}^0)\lambda_2^{\mathbf{b}}(z^0, r/())}.$$
(42)

We put in Theorem 11  $R = \frac{r}{2\lambda_2^{\mathbf{b}}(r)}$ . Then there exist  $P_2 = P_2\left(\frac{r}{2\lambda_2^{\mathbf{b}}(r)}\right) \ge 1$  and  $\eta\left(\frac{r}{2\lambda_2^{\mathbf{b}}(r)}\right) \in \left(0, \frac{r}{2\lambda_2^{\mathbf{b}}(r)}\right)$  that for all  $z^0 \in \mathbb{B}_n$ ,  $t_0 \in S_{z^0}$  and some  $r^* = r^*(z^0, t_0) \in \left[\eta\left(\frac{r}{2\lambda_2^{\mathbf{b}}(r)}\right), \frac{r}{2\lambda_2^{\mathbf{b}}(r)}\right]$  an inequality (28) holds with  $r^*$  instead of r. Using Cauchy inequality, we get

$$\frac{1}{m!} \left| \frac{\partial^m F(z^0 + t_0 \mathbf{b})}{\partial \mathbf{b}^m} \right| \le \left( \frac{L(z^0 + t_0 \mathbf{b})}{r^*} \right)^m \max \left\{ |F(z^0 + t \mathbf{b})| : |t - t_0| = \frac{r^*}{L(z^0 + t_0 \mathbf{b})} \right\} \le \\
\le P_2 \left( \frac{L(z^0 + t_0 \mathbf{b})}{\eta} \right)^m \min \left\{ |F(z^0 + t \mathbf{b})| : |t - t_0| = \frac{r^*}{L(z^0 + t_0 \mathbf{b})} \right\}.$$

From (42) for every  $z^0 \in \mathbb{B}_n \backslash G_r^{\mathbf{b}}(F)$  the set

$$\left\{ z^0 + t\mathbf{b} : |t - t_0| \le \frac{r}{2\lambda_2^{\mathbf{b}}(r)L(z^0 + t_0\mathbf{b})} \right\}$$

does not contain zeros of function  $F(z^0 + t\mathbf{b})$ . Therefore, applying to  $\frac{1}{F(z^0 + t\mathbf{b})}$  a maximum modulus principle in variable  $t \in S_{z^0}$ , we have

$$|F(z^0 + t_0 \mathbf{b})| \ge \min \left\{ |F(z^0 + t\mathbf{b})| : |t - t_0| = \frac{r^*}{L(z^0 + t_0 \mathbf{b})} \right\}.$$

Thus,

$$\left| \frac{\partial^m F(z^0 + t_0 \mathbf{b})}{\partial \mathbf{b}^m} \right| \le m! \frac{P_2}{\eta^m} L^m(z^0 + t_0 \mathbf{b}) |F(z^0 + t_0 \mathbf{b})|.$$

Hence, we proved a needed inequality with  $P = P_2 m! \eta^{-m}$ .

Using Lemma 6, we deduce a following theorem.

**Theorem 13.** Let  $\beta > 1$ ,  $L \in Q_{\mathbf{b},\beta}(\mathbb{B}_n)$ ,  $g_0(z), \ldots, g_p(z), h(z)$  be analytic in  $\mathbb{B}_n$  functions of bounded L-index in direction  $\mathbf{b}$ ,  $\mathbb{B}_n \backslash G^{\mathbf{b}}_{\beta}(g_0) \neq \emptyset$  and for every  $r \in (0; \beta]$  there exists T = T(r) > 0 that for each  $z \in \mathbb{B}_n \backslash G^{\mathbf{b}}_r(g_0)$  and  $j = 1, \ldots, p$  inequality holds

$$|g_j(z)| \le TL^j(z)|g_0(z)|.$$
 (43)

Then an analytic function F(z),  $z \in \mathbb{B}_n$ , which satisfies an equation (41), is of bounded L-index in direction **b**.

*Proof.* For every given  $z^0 \in \mathbb{B}_n$  let  $b_k^0$  be zeros of function  $g_0(z^0 + t\mathbf{b})$  and  $\{c_k^0\}$  be a set of zeros of all functions  $g_0(z^0 + t\mathbf{b})$ ,  $g_1(z^0 + t\mathbf{b})$ , ...,  $g_p(z^0 + t\mathbf{b})$  and  $h(z^0 + t\mathbf{b})$ , as functions of one variable  $t \in S_{z^0}$ . Obviously, this inclusion is valid  $\{b_k^0\} \subset \{c_k^0\}$ . We put

$$G_r^{\mathbf{b}}(z^0) = \bigcup_k \left\{ z^0 + t\mathbf{b} : |t - c_k^0| \le \frac{r}{L(z^0 + c_k^0 \mathbf{b})} \right\}, \ G_r^{\mathbf{b}} = \bigcup_{z^0} G_r^{\mathbf{b}}(z^0).$$

It is easy to see that  $G_r^{\mathbf{b}} = G_r^{\mathbf{b}}(h) \cup \bigcup_{j=1}^p G_r^{\mathbf{b}}(g_j)$ . Suppose that  $\mathbb{B}_n \setminus G_r^{\mathbf{b}}(g_0) \neq \emptyset$ . Lemma 6 and equation (43) implies that for every  $r \in (0, \beta]$  there exists  $T^* = T^*(r) > 0$  such that for all  $z \in \mathbb{B}_n \setminus G_r^{\mathbf{b}}$  the following inequalities hold

$$\left| \frac{\partial h(z)}{\partial \mathbf{b}} \right| \le T^* |h(z)| L(z), |g_j(z)| \le T^* |g_0(z)| L^j(z), \ j \in \{1, 2, \dots, p, \}$$

$$\left| \frac{\partial g_j(z)}{\partial \mathbf{b}} \right| \le P(r)L(z)|g_j(z)| \le T^*(r)|g_0(z)|L^{j+1}(z), \ j \in \{0, 1, 2, \dots, p, \}.$$

In equation (41) we evaluate a derivative in direction  $\mathbf{b}$ :

$$g_0(z)\frac{\partial^{p+1}F(z)}{\partial \mathbf{b}^{p+1}} + \sum_{j=1}^p g_j(z)\frac{\partial^{p+1-j}F(z)}{\partial \mathbf{b}^{p+1-j}} + \sum_{j=0}^n \frac{\partial g_j(z)}{\partial \mathbf{b}}\frac{\partial^{p-j}F(z)}{\partial \mathbf{b}^{p-j}} = \frac{\partial h(z)}{\partial \mathbf{b}}.$$

This obtained equality implies that for all  $z \in \mathbb{B}_n \setminus G_r^{\mathbf{b}}$ :

$$|g_{0}(z)| \left| \frac{\partial^{p+1} F(z)}{\partial \mathbf{b}^{p+1}} \right| \leq \left| \frac{\partial h(z)}{\partial \mathbf{b}} \right| + \sum_{j=1}^{p} |g_{j}(z)| \left| \frac{\partial^{p+1-j} F(z)}{\partial \mathbf{b}^{p+1-j}} \right| +$$

$$+ \sum_{j=0}^{p} \left| \frac{\partial g_{j}(z)}{\partial \mathbf{b}} \right| \left| \frac{\partial^{p-j} F(z)}{\partial \mathbf{b}^{p-j}} \right| \leq T^{*} |h(z)| L(z) + \sum_{j=1}^{p} |g_{j}(z)| \left| \frac{\partial^{p+1-j} F(z)}{\partial \mathbf{b}^{p+1-j}} \right| +$$

$$+ \sum_{j=0}^{p} \left| \frac{\partial g_{j}(z)}{\partial \mathbf{b}} \right| \left| \frac{\partial^{p-j} F(z)}{\partial \mathbf{b}^{p-j}} \right| \leq T^{*} L(z) \sum_{j=0}^{p} |g_{j}(z)| \left| \frac{\partial^{p-j} F(z)}{\partial \mathbf{b}^{p-j}} \right| +$$

$$+ \sum_{j=1}^{p} |g_{j}(z)| \left| \frac{\partial^{p+1-j} F(z)}{\partial \mathbf{b}^{p+1-j}} \right| + \sum_{j=0}^{p} \left| \frac{\partial g_{j}(z)}{\partial \mathbf{b}} \right| \left| \frac{\partial^{p-j} F(z)}{\partial \mathbf{b}^{p-j}} \right| \leq$$

$$\leq T^*|g_0(z)| \left( T^*L(z) \sum_{j=0}^p L^j(z) \left| \frac{\partial^{p-j} F(z)}{\partial \mathbf{b}^{p-j}} \right| + \sum_{j=1}^p L^j(z) \left| \frac{\partial^{p+1-j} F(z)}{\partial \mathbf{b}^{p+1-j}} \right| + \sum_{j=0}^p L^{j+1}(z) \left| \frac{\partial^{p-j} F(z)}{\partial \mathbf{b}^{p-j}} \right| \right) = T^*|g_0(z)|L^{p+1}(z)| \left( (T^* + 1) \times \sum_{j=0}^p \frac{1}{L^{p-j}(z)} \left| \frac{\partial^{p-j} F(z)}{\partial \mathbf{b}^{p-j}} \right| + \sum_{j=1}^p \frac{1}{L^{p+1-j}(z)} \left| \frac{\partial^{p+1-j} F(z)}{\partial \mathbf{b}^{p+1-j}} \right| \right) \leq$$

$$\leq T^*((T^* + 1)(p+1) + p)|g_0(z)|L^{p+1}(z) \max \left\{ \frac{1}{L^j(z)} \left| \frac{\partial^j F(z)}{\partial \mathbf{b}^j} \right| : 0 \leq j \leq p \right\}.$$

Thus, for every r > 0 there exists  $P_3 = P_3(r) > 0$  that for all  $z \in \mathbb{B}_n \setminus G_r^{\mathbf{b}}$  inequality holds

$$\frac{1}{L^{p+1}(z)} \left| \frac{\partial^{p+1} F(z)}{\partial \mathbf{b}^{p+1}} \right| \le P_3 \max \left\{ \frac{1}{L^j(z)} \left| \frac{\partial^j F(z)}{\partial \mathbf{b}^j} \right| : 0 \le j \le p \right\}.$$
(44)

Let  $z^0 + t_0 \mathbf{b}$  be an arbitrary point with  $\mathbb{B}_n$  and

$$K^{0} = \left\{ z^{0} + t_{0}\mathbf{b} : |t - t_{0}| \le \frac{\beta}{L(z^{0} + t_{0}\mathbf{b})} \right\}.$$

But  $g_0, g_1, \ldots, g_p, h$  are analytic in  $\mathbb{B}_n$  functions of bounded L-index in direction **b**. Hence, by Theorem 12 the set  $K^0$  contains at most  $N < +\infty$  elements of the set  $\{c_k^0\}$  and N is independent of  $z^0$  and  $t_0$ .

Let  $\widetilde{K}_k^0 = \left\{ z^0 + t\mathbf{b} : |t - c_k^0| \le \frac{\lambda_1^{\mathbf{b}}(\beta)(\beta - 1)}{8(N+1)L(z^0 + c_k^0\mathbf{b})} \right\}$ . From condition  $L \in Q_{\mathbf{b}\beta}(\mathbb{B}_n)$  it follows  $L(z^0 + c_k^0\mathbf{b}) \ge \lambda_1^{\mathbf{b}}(1)L(z^0 + t_0\mathbf{b})$ . If  $c_k^0 \in K^0$  then  $\widetilde{K}_k^0$  is a subset  $K_k^0$ 

$$\widetilde{K}_k^0 \subset K_k^0 = \left\{ z^0 + t\mathbf{b} : |t - c_k^0| \le \frac{\beta - 1}{8(N+1)L(z^0 + t_0\mathbf{b})} \right\}.$$

From the presented considerations, we deduce that for  $z^0 + t\mathbf{b} \in K^0 \setminus \bigcup_{c_k^0 \in K^0} K_k^0$  the inequality (44) holds with  $P_3 = P_3\left(\frac{\lambda_1^{\mathbf{b}}(\beta)(\beta-1)}{8(N+1)}\right)$ .

Again for these  $z^0 + t\mathbf{b} \in K^0 \setminus \bigcup_{c_k^0 \in K^0} K_k^0$  inequality holds  $L(z^0 + t_0\mathbf{b}) \ge \frac{L(z^0 + t\mathbf{b})}{\lambda_2^{\mathbf{b}}(\beta)}$ . Using (44), we have

$$\frac{1}{L^{p+1}(z^{0}+t_{0}\mathbf{b})} \left| \frac{\partial^{p+1}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{p+1}} \right| \leq \lambda_{2}^{\mathbf{b}}(\beta) \frac{1}{L^{p+1}(z^{0}+t\mathbf{b})} \times \\
\times \left| \frac{\partial^{p+1}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{p+1}} \right| \leq P_{3}(\lambda_{2}^{\mathbf{b}}(\beta))^{p+1} \max \left\{ \frac{1}{L^{j}(z^{0}+t\mathbf{b})} \left| \frac{\partial^{j}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{j}} \right| : \\
0 \leq j \leq p \right\} \leq P_{3}(\lambda_{2}^{\mathbf{b}}(\beta))^{p+1} \max \left\{ \frac{1}{L^{j}(z^{0}+t_{0}\mathbf{b})} \left| \frac{\partial^{j}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{j}} \right| \times \\
\times \left( \frac{1}{\lambda_{1}^{\mathbf{b}}(\beta)} \right)^{j} : 0 \leq j \leq p \right\} \leq P_{3} \left( \frac{\lambda_{2}^{\mathbf{b}}(\beta)}{\lambda_{1}^{\mathbf{b}}(\beta)} \right)^{p} \lambda_{1}^{\mathbf{b}}(\beta) \max \left\{ \frac{1}{L^{j}(z^{0}+t_{0}\mathbf{b})} \times \right. \\
\times \left| \frac{\partial^{j}F(z^{0}+t\mathbf{b})}{\partial \mathbf{b}^{j}} \right| : 0 \leq j \leq p \right\} = P_{4}g_{z^{0}}(t_{0},t), \tag{45}$$

where  $P_4 = P_3 \lambda_2^{\mathbf{b}}(\beta) \left( \frac{\lambda_2^{\mathbf{b}}(\beta)}{\lambda_1^{\mathbf{b}}(\beta)} \right)^p$  and

$$g_{z^0}(t_0, t) = \max \left\{ \frac{1}{L^j(z^0 + t_0 \mathbf{b})} \left| \frac{\partial^j F(z^0 + t \mathbf{b})}{\partial \mathbf{b}^j} \right| : 0 \le j \le p \right\}.$$

Let D be a sum of diameters of sets  $K_k^0$ . Then

$$D \le \frac{2|\mathbf{b}|(\beta - 1)N}{8(N+1)L(z^0 + t_0\mathbf{b})} \le \frac{|\mathbf{b}|(\beta - 1)}{4L(z^0 + t_0\mathbf{b})}.$$

Therefore, there exist radii  $r_1 \in \left[\frac{\beta}{4}, \frac{\beta}{2}\right]$  and  $r_2 \in \left[\frac{\beta+1}{2}, \beta\right]$  with property: if either

$$z^{0} + t\mathbf{b} \in C_{1} = \left\{ z^{0} + t\mathbf{b} : |t - t_{0}| = \frac{r_{1}}{L(z^{0} + t_{0}\mathbf{b})} \right\}$$

or 
$$z^0 + t\mathbf{b} \in C_2 = \left\{ z^0 + t\mathbf{b} : |t - t_0| = \frac{r_2}{L(z^0 + t_0\mathbf{b})} \right\}$$

then  $z^0 + t\mathbf{b} \in K^0 \setminus \bigcup_{c_k^0 \in K^0} K_m^0$ . We take two any points  $z^0 + t_1\mathbf{b} \in C_1$  and  $z^0 + t_2\mathbf{b} \in C_2$  and connect them by a smooth curve  $\gamma = \{z^0 + t\mathbf{b} : t = t(s), 0 \le s \le T\}$  that  $F(z^0 + t(s)\mathbf{b}) \ne 0$  and  $\gamma \subset K^0 \setminus \bigcup_{c_k^0 \in K^0} K_m^0$ . This curve can be selected such that for its length the following estimate holds

$$|\gamma| \leq |\mathbf{b}| \left( \frac{\pi r_{1}}{L(z^{0} + t_{0}\mathbf{b})} + \frac{r_{2} - r_{1}}{L(z^{0} + t_{0}\mathbf{b})} + \frac{\pi N(\beta - 1)}{8(N + 1)L(z^{0} + t_{0}\mathbf{b})} \right) \leq$$

$$\leq |\mathbf{b}| \left( \frac{r_{2} + (\pi - 1)r_{1}}{L(z^{0} + t_{0}\mathbf{b})} + \frac{\pi(\beta - 1)}{8L(z^{0} + t_{0}\mathbf{b})} \right) \leq$$

$$\leq |\mathbf{b}| \frac{1}{L(z^{0} + t_{0}\mathbf{b})} \left( \frac{(\pi - 1)\beta}{2} + \beta + \frac{\pi(\beta - 1)}{8} \right) < \frac{3\pi\beta|\mathbf{b}|}{L(z^{0} + t_{0}\mathbf{b})}.$$
(46)

Then an inequality (45) holds on  $\gamma$  that is

$$\frac{1}{L^{p+1}(z^0 + t_0 \mathbf{b})} \left| \frac{\partial^{p+1} F(z^0 + t(s) \mathbf{b})}{\partial \mathbf{b}^{p+1}} \right| \le P_4 g_{z^0}(t_0, t(s)), \ 0 \le s \le T.$$

In the proof of Theorem 10 we obtained that the function  $g_{z^0}(t_0, t(s))$  is continuous on [0, T] and continuously differentiable except, perhaps, finite number of points. Besides, for complex-valued function of real variable inequality holds  $\frac{d}{ds}|\varphi(s)| \leq \left|\frac{d}{ds}\varphi(s)\right|$  except the points, where  $\varphi(s) = 0$ .

Then, in view of (45), we have

$$\frac{d}{ds}g_{z^0}(t_0, t(s)) \le \max\left\{\frac{d}{ds}\frac{1}{L^j(z^0 + t_0\mathbf{b})} \left| \frac{\partial^j F(z^0 + t(s)\mathbf{b})}{\partial \mathbf{b}^j} \right| : 0 \le j \le p \right\} \le \\
\le \max\left\{\frac{1}{L^{j+1}(z^0 + t_0\mathbf{b})} \left| \frac{\partial^{j+1} F(z^0 + t(s)\mathbf{b})}{\partial \mathbf{b}^{j+1}} \right| |t'(s)| L(z^0 + t_0\mathbf{b}) : 0 \le j \le p \right\} \le$$

$$\leq \max \left\{ \frac{1}{L^{j+1}(z^{0} + t_{0}\mathbf{b})} \left| \frac{\partial^{j+1} F(z^{0} + t(s)\mathbf{b})}{\partial \mathbf{b}^{j+1}} \right| : 0 \leq j \leq p; \left| \frac{\partial^{p+1} F(z^{0} + t(s)\mathbf{b})}{\partial \mathbf{b}^{p+1}} \right| \right\} \times \\ \times |t'(s)| L(z^{0} + t_{0}\mathbf{b}) \leq P_{5} g_{z^{0}}(t_{0}, t(s)) |t'(s)| L(z^{0} + t_{0}\mathbf{b}).$$

where  $P_5 = \max\{1, P_4\}$ . But (46) is true, then

$$\left| \ln \frac{g_{z^0}(t_0, t_2)}{g_{z^0}(t_0, t_1)} \right| = \left| \int_0^T \frac{1}{g_{z^0}(t_0, t(s))} \frac{d}{ds} g_{z^0}(t_0, t(s)) ds \right| \le$$

$$\le P_5 L(z^0 + t_0 \mathbf{b}) \int_0^T |t'(s)| ds \le P_5 L(z^0 + t_0 \mathbf{b}) |\gamma| \le 3\pi \beta |\mathbf{b}| P_5,$$

i.e.

$$g_{z^0}(t_0, t_2) \le g_{z^0}(t_0, t_1) \exp\{3\pi\beta |\mathbf{b}| P_5\}.$$

We can choose  $t_2$  that  $|F(z^0 + t_2\mathbf{b})| = \max\{|F(z^0 + t\mathbf{b})| : z^0 + t\mathbf{b} \in C_2\}$ . Hence,

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\beta + 1}{2L(z^{0} + t_{0}\mathbf{b})} \right\} \le |F(z^{0} + t_{2}\mathbf{b})| \le$$

$$\le g_{z^{0}}(t_{0}, t_{2}) \le g_{z^{0}}(t_{0}, t_{1}) \exp\{3\pi\beta|\mathbf{b}|P_{5}\}. \tag{47}$$

Since  $z^0 + t_1 \mathbf{b} \in C_1$  we apply Cauchy inequality in variable t for all j = 1, 2, ..., p, and obtain

$$\left| \frac{\partial^{j} F(z^{0} + t_{1} \mathbf{b})}{\partial \mathbf{b}^{j}} \right| \leq j! \left( 10L(z^{0} + t_{0} \mathbf{b}) \right)^{j} \max \left\{ |F(z^{0} + t \mathbf{b})| : |t - t_{1}| = \frac{1}{2\beta L(z^{0} + t_{0} \mathbf{b})} \right\} \leq g! \left( 10L(z^{0} + t_{0} \mathbf{b}) \right)^{j} \max \left\{ |F(z^{0} + t \mathbf{b})| : |t - t_{0}| = \frac{1}{\beta L(z^{0} + t_{0} \mathbf{b})} \right\}$$

It follows that

$$g_{z^0}(t_0, t_1) \le p! 10^p \max \left\{ |F(z^0 + t\mathbf{b})| : |t - t_0| = \frac{1}{\beta L(z^0 + t_0\mathbf{b})} \right\}$$
 (48)

From inequalities (47) and (48) we have

$$\max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{\beta + 1}{2L(z^{0} + t_{0}\mathbf{b})} \right\} \le p! 10^{p} \exp\{|\mathbf{b}|P_{5}\} \times \max \left\{ |F(z^{0} + t\mathbf{b})| : |t - t_{0}| = \frac{1}{\beta L(z^{0} + t_{0}\mathbf{b})} \right\}.$$

Therefore, by Theorem 9 an analytic function F(z) is of bounded L-index in direction b.

### 8 Growth of analytic in $\mathbb{B}_n$ functions of bounded L-index in the direction.

We denote  $a^+ = \max\{a, 0\}$ .

**Theorem 14.** Let  $L: \mathbb{B}_n \to \mathbb{R}_+$ , for every  $z^0 \in \mathbb{B}_n$ ,  $\theta \in [0, 2\pi]$  a function  $L(z^0 + re^{i\theta}\mathbf{b})$  be a continuously differentiable function of real variable  $r \in [0, R)$ , where  $R = \min\{t \in \mathbb{R}_+ : |z^0 + te^{i\theta}\mathbf{b}| = 1\}$ . If an analytic in  $\mathbb{B}_n$  function F is of bounded L-index in direction  $\mathbf{b}$  then for every  $z^0 \in \mathbb{B}_n$ ,  $\theta \in [0, 2\pi]$ ,  $r \in [0, R)$  and every integer  $p \geq 0$ 

$$\ln\left(\frac{1}{p!L^{p}(z^{0}+re^{i\theta}\mathbf{b})}\left|\frac{\partial^{p}F(z^{0}+re^{i\theta}\mathbf{b})}{\partial\mathbf{b}^{p}}\right|\right) \leq \ln\max\left\{\frac{1}{k!L^{k}(z^{0})}\left|\frac{\partial^{k}F(z^{0})}{\partial\mathbf{b}^{k}}\right|:0\leq k\leq N\right\} + \int_{0}^{r}\left\{(N+1)L(z^{0}+te^{i\theta}\mathbf{b})+N\frac{(-L'_{t}(z^{0}+te^{i\theta}\mathbf{b}))^{+}}{L(z^{0}+te^{i\theta}\mathbf{b})}\right\}dt$$

$$(49)$$

But if, in addition, for every  $z^0 \in \mathbb{B}_n$  and  $\theta \in [0, 2\pi] \left( -\frac{\partial L(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}} \right)^+ / (L^2(z^0 + re^{i\theta}\mathbf{b})) \Rightarrow 0$  as  $|z^0 + re^{i\theta}\mathbf{b}| \to 1$  then for every  $z^0 \in \mathbb{B}_n$  and  $\theta \in [0, 2\pi]$ 

$$\frac{\overline{\lim}}{|z^0 + re^{i\theta}\mathbf{b}| \to 1} \frac{\ln |F(z^0 + re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})dt} \le N_{\mathbf{b}}(F, L) + 1,$$
(50)

holds.

Proof. We remark that  $R \geq \frac{1-|z^0|}{|\mathbf{b}|}$ , because  $|z^0 + te^{i\theta}\mathbf{b}| \leq |z^0| + |t| \cdot |\mathbf{b}| \leq |z^0| + \frac{1-|z^0|}{|\mathbf{b}|} \cdot |\mathbf{b}| \leq 1$ . The condition  $r \in [0, R)$  provides  $z^0 + re^{i\theta}\mathbf{b} \in \mathbb{B}_n$ .

Denote  $N = N_{\mathbf{b}}(F, L)$ . For fixed  $z^0 \in \mathbb{B}_n$  and  $\theta \in [0, 2\pi]$  we consider the function

$$g(r) = \max \left\{ \frac{1}{k! L^k(z^0 + re^{i\theta}\mathbf{b})} \left| \frac{\partial^k F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k} \right| : 0 \le k \le N \right\}.$$
 (51)

Since the function  $\frac{1}{k!L^k(z^0+re^{i\theta}\mathbf{b})}\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right|$  is a continuously differentiable of real  $r \in [0,R)$ , the function g is continuously differentiable on [0,R), exception, perhaps, a finite set of points, and

$$g'(r) \leq \max \left\{ \frac{d}{dt} \left( \frac{1}{k!L^k(z^0 + re^{i\theta}\mathbf{b})} \left| \frac{\partial^k F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k} \right| \right) : 0 \leq k \leq N \right\} \leq$$

$$\leq \max \left\{ \frac{1}{k!L^k(z^0 + re^{i\theta}\mathbf{b})} \left| \frac{\partial^{k+1} F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^{k+1}} \right| - \frac{1}{k!L^k(z^0 + re^{i\theta}\mathbf{b})} \times \right.$$

$$\times \left| \frac{\partial^k F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k} \right| k \frac{L'_r(z^0 + re^{i\theta}\mathbf{b})}{L(z^0 + re^{i\theta}\mathbf{b})} : 0 \leq k \leq N \right\} \leq$$

$$\leq \max \left\{ \frac{1}{(k+1)!L^{k+1}(z^0 + re^{i\theta}\mathbf{b})} \left| \frac{\partial^{k+1} F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^{k+1}} \right| (k+1)L(z^0 + re^{i\theta}\mathbf{b}) + \right.$$

$$\left. + \frac{1}{k!L^k(z^0 + re^{i\theta}\mathbf{b})} \left| \frac{\partial^k F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k} \right| k \frac{(-L'_r(z^0 + re^{i\theta}\mathbf{b}))^+}{L(z^0 + re^{i\theta}\mathbf{b})} :$$

$$0 \leq k \leq N \right\} \leq g(r) \left( (N+1)L(z^0 + re^{i\theta}\mathbf{b}) + N \frac{(-L'_r(z^0 + re^{i\theta}\mathbf{b}))^+}{L(z^0 + re^{i\theta}\mathbf{b})} \right).$$

Thus, we have

$$\frac{d}{dr}\ln g(r) \le (N+1)L(z^0 + re^{i\theta}\mathbf{b}) + N\frac{(-L'_r(z^0 + re^{i\theta}\mathbf{b}))^+}{L(z^0 + re^{i\theta}\mathbf{b})}.$$

Since F is a function of bounded L-index in the direction then  $g(0) \neq 0$  and

$$g(r) \le g(0) \exp \left\{ \int_0^r \left( (N+1)L(z^0 + te^{i\theta}\mathbf{b}) + N \frac{(-L'_t(z^0 + te^{i\theta}\mathbf{b}))^+}{L(z^0 + te^{i\theta}\mathbf{b})} \right) dt \right\}, \ r \to R,$$

so that

$$\ln g(r) \le \ln g(0) + \int_0^r \left( (N+1)L(z^0 + te^{i\theta}\mathbf{b}) + N \frac{(-L_t'(z^0 + te^{i\theta}\mathbf{b}))^+}{L(z^0 + te^{i\theta}\mathbf{b})} \right) dt, \ r \to R.$$

Using (51), we obtain (49). If, in addition, for every  $z^0 \in \mathbb{B}_n$  and  $\theta \in [0, 2\pi] \left( -\frac{\partial L(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}} \right)^+ / (L^2(z^0 + re^{i\theta}\mathbf{b})) \implies 0$  when  $|z^0 + re^{i\theta}\mathbf{b}| \to 1$  then

$$g(r) \le g(0) \exp\left\{ (N+1) \int_0^r \left( L(z^0 + te^{i\theta} \mathbf{b}) + \frac{(-L'_r(z^0 + te^{i\theta} \mathbf{b}))^+}{L(z^0 + te^{i\theta} \mathbf{b})} \right) dt \right\} =$$

$$= g(0) \exp\left\{ (N+1)(1+o(1)) \int_0^r L(z^0 + te^{i\theta} \mathbf{b}) dt \right\}, \ r \to R,$$

so that

$$|F(z^0 + re^{i\theta}\mathbf{b})| \le g(r) \le g(0) \exp\left\{ (N+1)(1+o(1)) \int_0^r L(z^0 + te^{i\theta}\mathbf{b}) dt \right\}, \ r \to R,$$

for  $\theta \in [0, 2\pi], z^0 \in \mathbb{B}_n$ , whence

$$\ln|F(z^{0} + re^{i\theta}\mathbf{b})| \le g(0) + (N+1)(1+o(1)) \int_{0}^{r} L(z^{0} + te^{i\theta}\mathbf{b})dt, \ r \to R.$$
 (52)

Moreover, for every  $z^0 \in \mathbb{B}_n$  and  $\theta \in [0, 2\pi]$  we have

$$\overline{\lim_{|z^0+re^{i\theta}\mathbf{b}|\to 1}} \frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt} \le N_{\mathbf{b}}(F,L) + 1.$$

Remark 5. The equations (49) and (50) can be written in more convenient forms:

$$\ln \max_{|t|=r} \left( \frac{1}{p! L^p(z^0 + t\mathbf{b})} \left| \frac{\partial^p F(z^0 + t\mathbf{b})}{\partial \mathbf{b}^p} \right| \right) \le \ln \max \left\{ \frac{1}{k! L^k(z^0)} \left| \frac{\partial^k F(z^0)}{\partial \mathbf{b}^k} \right| : 0 \le k \le N \right\} + \\
+ \max_{\theta \in [0, 2\pi]} \int_0^r \left\{ (N+1) L(z^0 + te^{i\theta}\mathbf{b}) + N \frac{(-L'_t(z^0 + te^{i\theta}\mathbf{b}))^+}{L(z^0 + te^{i\theta}\mathbf{b})} \right\} dt \tag{53}$$

and

$$\frac{\overline{\lim}}{|z^0 + re^{i\theta}\mathbf{b}| \to 1} \max_{\theta \in [0, 2\pi]} \frac{\ln |F(z^0 + re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})dt} \le N_{\mathbf{b}}(F, L) + 1.$$
(54)

Besides, if we put  $z^0 = 0$  then the estimate (52) implies a following inequality

$$\frac{\overline{\lim}_{R \to 1/|\mathbf{b}|} \frac{\ln \max\{|F(t\mathbf{b})| : |t| = R\}}{\max_{\theta \in [0,2\pi]} \int_0^R L(re^{i\theta}\mathbf{b}) dr} \le N_{\mathbf{b}}(F, L) + 1.$$
(55)

For n = 1 we deduce corollaries.

Corollary 2. Let  $l: \mathbb{D} \to \mathbb{R}_+$ ,  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and for  $\theta \in [0, 2\pi]$  a function  $l(re^{i\theta})$  be a continuously differentiable function of real variable  $t \in [0, 1)$ . If f(z) is an analytic function of bounded l-index then for every integer  $p \geq 0$ 

$$\ln \frac{|f^{(p)}(re^{i\theta})|}{p!l^{p}(re^{i\theta})} \leq 
\leq \ln \max \left\{ \frac{|f^{(k)}(0)|}{k!l^{k}(0)} : 0 \leq k \leq N \right\} + \int_{0}^{r} \left\{ (N+1)l(te^{i\theta}) + N \frac{(-L'_{t}(te^{i\theta}))^{+}}{L(te^{i\theta})} \right\} dt$$
(56)

If, in addition,  $(-l'(re^{i\theta}))^+/l^2(re^{i\theta}) \Rightarrow 0$  as  $r \to 1$  then

$$\overline{\lim_{r \to 1}} \frac{\ln |f(re^{i\theta})|}{\int_0^r l(te^{i\theta})dt} \le N(f,l) + 1, \ \theta \in [0,2\pi]$$

$$(57)$$

holds, where N(f, l) is l-index of function f.

**Remark 6.** The equations (56) and (57) can be written in more convenient forms

$$\ln \max_{|t|=r} \frac{|f^{(p)}(t)|}{p!l^{p}(t)} \leq \\
\leq \ln \max \left\{ \frac{|f^{(p)}(0)|}{p!l^{p}(0)} : 0 \leq k \leq N \right\} + \max_{\theta \in [0,2\pi]} \int_{0}^{r} \left\{ (N+1)l(te^{i\theta}) + N \frac{(-L'_{t}(te^{i\theta}))^{+}}{L(te^{i\theta})} \right\} dt \tag{58}$$

and

$$\overline{\lim_{r \to 1}} \max_{\theta \in [0,2\pi]} \frac{\ln |f(re^{i\theta})|}{\int_0^r l(te^{i\theta})dt} \le N(f,l) + 1,$$
(59)

The Corollary 2 is an improvement of similar result of Sheremeta and Strochyk [27] because we do not assume that l = l(|z|).

Corollary 3. Let  $F: \mathbb{B}_n \to \mathbb{C}$  be an analytic function of bounded L-index in direction  $\mathbf{b}$ ,  $N = N_{\mathbf{b}}(F, L)$ ,  $z^0$  be a fixed point in  $\mathbb{B}_n$  that  $F(z^0) = 1$ . Then for every  $r \in [0, R)$ , where  $R = \min\{t \in \mathbb{R}_+ : |z^0 + te^{i\theta}\mathbf{b}| = 1\}$ , the next inequality

$$\int_0^r \frac{n(t, z^0, 0, 1/F)}{t} dt \le \ln \max\{|F(z^0 + t\mathbf{b})| : |t| = r\} \le$$

$$\le \ln \max\left\{\frac{1}{p!L^p(z^0)} \left| \frac{\partial^p F(z^0)}{\partial \mathbf{b}^p} \right| : 0 \le k \le N\right\} +$$

$$+ \max_{\theta \in [0, 2\pi]} \int_0^r \left\{ (N+1)L(z^0 + te^{i\theta}\mathbf{b}) + N \frac{(-L'_t(z^0 + te^{i\theta}\mathbf{b}))^+}{L(z^0 + te^{i\theta}\mathbf{b})} \right\} dt$$

holds.

*Proof.* We consider a function  $F(z^0 + t\mathbf{b})$  as a function of one variable t. Thus, the first inequality follows from the classical Jensen Theorem. In addition, the second inequality follows from (53) for p = 0.

#### References

- [1] T. O. Banakh and V. O. Kushnir, On growth and distribution of zeros of analytic functions of bounded l-index in arbitrary domains, Mat. Stud. 14 (2000), no. 2, 165–170. http://matstud.org.ua/texts/2000/14\_2/14\_2\_165-170.pdf
- [2] A. I. Bandura, Sufficient conditions of boundedness L-index in direction for entire functions with "plane" zeros of genus p, Mat. Visn. Nauk. Tov. Im. Shevchenka. 6 (2009), 44–49. (Ukrainian) https://www.researchgate.net/publication/262916755\_i\_L-\_\_\_\_p\_\_\_
- [3] A. I. Bandura, A modified criterion of boundedness of L-index in direction, Mat. Stud. **39** (2013), no. 1, 99–102. http://matstud.org.ua/texts/2013/39\_1/99-102.pdf
- Banduraand0. В. Skaskiv, Entire functions of bounded Lindex direction. Mat. Stud. (2007),no. 1, 30-52.(Ukrainian) http://matstud.org.ua/texts/2007/27\_1/27\_1\_030\_052.pdf
- [5] A. I. Bandura and O. B. Skaskiv, Entire functions of bounded and unbounded index in direction, Mat. Stud. 27 (2007), no. 2, 211–215. (Ukrainian) http://matstud.org.ua/texts/2007/27\_2/27\_2\_211\_215.pdf
- [6] A. I. Bandura and O. B. Skaskiv, Sufficient sets for boundedness L-index in direction for entire functions, Mat. Stud. **30** (2008), no. 2, 177–182. http://matstud.org.ua/texts/2008/30\_2/177-182.pdf
- [7] A. I. Bandura and O. B. Skaskiv, Boundedness of L-index in direction of functions of the form  $f(\langle z, m \rangle)$  and existence theorems, Mat. Stud. **41** (2014), no. 1, 45–52. http://matstud.org.ua/texts/2014/41\_1/45-52.pdf
- [8] M. T. Bordulyak and M. M. Sheremeta, Boundedness of the L-index of an entire function of several variables, Dopov. Akad. Nauk Ukr. 9 (1993), 10–13. (Ukrainian)
- [9] M. T. Bordulyak, The space of entire in  $\mathbb{C}^n$  functions of bounded L-index, Mat. Stud. 4 (1995), 53-58. (Ukrainian) http://matstud.org.ua/texts/1995/4/4\_053-058.pdf
- [10] B. C. Chakraborty and Rita Chanda, A class of entire functions of bounded index in several variables. J. Pure Math. 12 (1995), 16–21. MR1454732 (98h:32004)
- [11] B. C. Chakraborty and Tapas Kumar Samanta, On entire functions of bounded index in several variables. J. Pure Math. 17 (2000), 53–71. MR1840581 (2002f:32002)
- [12] B. C. Chakraborty and Tapas Kumar Samanta, On entire functions of L-bounded index. J. Pure Math. 18 (2001), 53–64. MR1916624 (2003e:30044)

- [13] W. K. Hayman, Differential equations and local valency. Pac. J. Math. 44 (1973), no. 1, 117-137. https://projecteuclid.org/euclid.pjm/1102948654
- [14] Gopala J. Krishna and S. M. Shah, Functions of bounded indices in one and several complex variables. MR0271345 (42 #6228) In: Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, Athens, Ohio, 1970, 223–235. MR271311 (42 #6194).
- [15] V. O. Kushnir and M. M. Sheremeta, Analytic functions of bounded l-index, Mat. Stud. 12 (1999), no. 1, 59-66. http://matstud.org.ua/texts/1999/12\_1/12\_1\_059-066.pdf
- [16] V. O. Kushnir, On analytic in disc function of bounded l-index, Visnyk Lviv Univ., Ser. Mech. & Math. 58 (2000), 21–24. (Ukrainian)
- [17] A. D. Kuzyk and M. M. Sheremeta, Entire functions of bounded l-distribution of values. Math. notes. **39** (1986), no. 1, 3–8. http://dx.doi.org/10.1007%2FBF01647624
- [18] *T. Lakshminarasimhan*, A note on entire functions of bounded index. J. Indian Math. Soc. **38** (1974), 43–49. *MR454011* (56 #12262).
- [19] B. Lepson,Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index. Proc. Sympos. Pure Vol.11, Amer. Math. Soc., Providence, R. Ι., (1968),298 - 307.http://www.ams.org/bookstore?fn=20&arg1=pspumseries&ikey=PSPUM-11 MR237788 (38 # 6069)
- [20] M. Salmassi, Functions of bounded indices in several variables, Indian J. Math. 31 (1989), no. 3, 249–257. MR1042643 (91d:32003)
- [21] S. M. Shah, Entire function of bounded index Lect. Notes in Math. 599 (1977), 117-145. http://dx.doi.org/10.1007/BFb0096833
- [22] M. Sheremeta, Analytic functions of bounded index, VNTL Publishers, Lviv, 1999, 141 p. http://www.vntl.com/en/?part=43&book=7
- Yu.S. |23| M.M. Sheremeta and Trukhan, Boundedness of  $_{
  m the}$ l-index of Naftalevich-Tsuji product, Ukr. Math. J. 56(2004),no. 305-317.http://dx.doi.org/10.1023/B:UKMA.0000036104.89729.3e
- [24] M.N. Sheremeta, Entire functions and Dirichlet series of bounded l-1992, 9, Vyssh. Uchebn. Zaved. Mat. no. 81 - 87(Russian) http://www.mathnet.ru/php/getFT.phtml?jrnid=ivm&paperid=4899&what=fullt&option\_lang= English transl. in Russian Math. (Iz. VUZ) 36 (1992), no. 9, 76–82. MR1233875 (95a:30025)

- [25] D. Soma sundaramandR. Thamizharasi, On entire functions of L-Indian J. Pure Appl. Math. 14 (1983), bounded index. no. 3, 330 - 344.http://insa.nic.in/writereaddata/UpLoadedFiles/IJPAM/20005a7c\_330.pdf
- [26] D. Somasundaram and R. Thamizharasi, A note on the entire functions of L-bounded index and L-type. Indian J. Pure Appl. Math. 19 (1988), no. 3, 284-294. http://insa.nic.in/writereaddata/UpLoadedFiles/IJPAM/20005a51\_284.pdf
- [27] S. N. Strochyk and Sheremeta M. M., Analytic in the unit disc functions of bounded index, Dopov. Akad. Nauk Ukr. 1 (1993), 19–22. (Ukrainian)
- [28] Yu. S. Trukhan, On the boundedness of l-index of Blaschke product, Visnyk Lviv Univ., Ser. Mech. & Math. 63 (2004), 143–147. (Ukrainian)
- [29] Yu. S. Trukhan, Preservation of l-index boundedness of the Blaschke product under zeros shifts, Mat. Stud. **25** (2006), no. 1, 29–37. (Ukrainian) http://matstud.org.ua/texts/2006/25\_1/25\_1\_029\_037.pdf
- [30] Yu. S. Trukhan and M. M. Sheremeta, On the boundedness l-index of a canonical product of zero genus and of a Blaschke product, Mat. Stud. 29 (2008), no. 1, 45–51. (Ukrainian) http://matstud.org.ua/texts/2008/29\_1/29\_1\_045\_051.pdf
- [31] Yu. S. Trukhan and M. M. Sheremeta, On l-index boundedness of the Blaschke product, Mat. Stud. 19 (2003), no. 1, 106–112. http://matstud.org.ua/texts/2003/19\_1/21\_30.pdf
- [32] Yu. S. Trukhan M. M. Boundedness *l*-index and Sheremeta, of Blaschke product, Mat. Stud. 17 (2002),2, 127 - 137.(Ukrainian) http://matstud.org.ua/texts/2002/17\_2/127\_137.pdf